1. Dummy entry.

PDP Math 1A *WS#2* *Fall 1987*

1. (review)

   (a) True or false: If $x < 0, |x| = -x$.
   Solve the following for x:

   (b) $0 < |x + 1| < 3$

   (c) $0 < |x - 2| < d,$ where $d > 0$.

   (d) $||x| - 1| < 1$

   (e) $|x - a| = x + a$

2. Miscellaneous questions from old Math 1A quizzes.

   (a) Determine the domain of $g(z) = [1 - (z^2 - 1)^{1/2}]^{1/2}$.

   (b) Find the domain, range, asymptotes, and symmetries of the relation $(y + 7)/(y - 7) = (x + 3)/(x - 3)$.

   (c) What are the asymptotes of $y = x/(|x| + 2x + 1)$?

   (d) Prove that the graph of $1 + x^2 + x^3 = y^2$ is not symmetric with respect to the origin.

   (e) Is the graph of $\{(x, y)|(2 + y^3)(x - 1) = (2 - y^3)(x + 1)\}$ symmetric with respect to the origin?

3. From a Fall 1981 midterm: Let S be the relation $\{(x, y)|(x^2 - 4)y = x^2 + 2x\}$.

   (a) Using set notation, describe the domain of S and the range of S.

   (b) Find all horizontal and vertical asymptotes of S and describe each by an equation.

   (c) Find numbers a and b such that the point (a, b) is on the graph of S but the point $(-a, b)$ is not. Then complete the following sentence: "The fact that (a, b) is on the graph but $(-a, b)$ is not shows that the graph of S is not symmetric about *????*."

4. For each of the following relations locate all asymptotes, determine the domain, range, and x- and y- intercepts, and draw a rough sketch of the relation's graph.

   (a) $y(x^2 - 1) = -1$

   (b) $y^2(x^2 - 1) = -1$

5. Give an example of a rational function that
   (a) has vertical asymptotes at $x = 0$ and $x = 4$ and a horizontal asymptote at $y = 1$.
   (b) has a vertical asymptote at $x = 1$, a horizontal asymptote at $y = 1$, and an x-intercept at 2.
   (c) has one horizontal asymptote and crosses this asymptote
      i. once
      ii. 3 times
      iii. n times
      iv. infinitely often

PDP Math 1A   *WS#3*   *Fall 1987*

1. Review
   (a) Let $f(x) = |x + 1| - 2$. Sketch

      i. $y = f(x)$, ii. $y = f(-x)$, iii. $y = -f(x)$, iv. $y = f(1/x)$

   (b) Find equations for the graphs sketched below. (Sorry: no pictures yet.)
2. Discuss the following relations. Find their domains, ranges, intercepts, and symmetries. Sketch.
   (a) $|x - 1| = |y + 2|$
   (b) $y = x/(1 - x^2)^{1/2}$
   (c) $4y^2 - 12xy + 8x^2 - 2x = 5$
3. Draw a quick sketch using the techniques that you learned in workshop.

$(a)\ y = (x + 1)/(x - 1)$	$(b)\ y = (x - 1)/(x + 1)$
$(c)\ y = (3x + 1)/(x - 2)$	$(d)\ y = (1 - x)/(2x + 3)$
$(e)\ y = 1/(x^2 + 4)$	$(f)\ y = 1/(x^2 - 4)$
$(g)\ y = 1/(x^2 + 2x + 2)$	$(h)\ y = 1/(x^2 - 3x + 2)$

4. (a) Find an equation for the set of points $P(x, y)$ which are equidistant from $(1, 3)$ and $(2, 4)$.
   (b) Find the distance from the point $(4, 3)$ to the line $y = 3x + 1$.
   (c) Find an equation for the set of points which are equidistant from the point $(3, 1)$ and the line $y = 2$.

5. Let L be a line with slope $m, m \neq 0$. Characterize geometrically all lines that have slope

(a) m (b) $-m$ (c) $-1/m$ (d) $1/m$

(For example, the answer to (a) is: all lines parallel to L.)

6. (a) Show that if a relation is symmetric both with respect to the y-axis and to the origin, then it is also symmetric with respect to the x-axis.
   (b) Let f be defined for all x in $\mathbf{R}$. Let $g(x) = [f(x) + f(-x)]/2$; $h(x) = [f(x) - f(-x)]/2$. Show that g is sym wrt the y-axis, h is symmetric wrt the origin, and that an arbitrary function can always be represented as the sum of two functions, one symmetric wrt the y-axis, the other, symmetric wrt the origin.

PDP Math 1A WS#4 Fall 1987

1. Let $f(x) = (x^2 - 1)/(x + 1)$
   (a) Sketch the graph of f, and determine its domain and range.
   (b) Evaluate $f(-1)$
   (c) Evaluate $f(+1)$
   (d) Evaluate $\lim_{x\to -1} f(x)$
   (e) Evaluate $\lim_{x\to +1} f(x)$

2. (a) For each of the following, sketch the graph and find $\lim_{x\to 2} f(x)$.
      i. $f(x) = (2x - 4)/(x - 2)$
      ii. $f(x) = (x^2 + x - 6)/(x - 2)$
      iii. $f(x) = (x - 2)/(x^2 - 4)$
      iv. $f(x) = (x - 2)/\sqrt{(x - 2)}$
   (b) Find a rational function g whose domain is $\{x | x \neq 1, 2\}$ and such that $g(x) = x^2$ wherever $g(x)$ is defined.

3. Calculate each of the following:

   a) $\lim_{x\to -1} |x|/x$ b) $\lim_{x\to 1} (x^3 - x^2 + x - 1)/(x^2 - 1)$

   c) $\lim_{x\to 0} \sqrt{x}$ d) $\lim_{x\to 0} [(1 + x)^2 - 1]/[(1 + x)^3 - 1]$

   e) $\lim_{x\to 2} (\sqrt{x} - \sqrt{2})/(x - 2)$

   f) $\lim_{x\to 3} f(x)$, where $f(x) = \begin{cases} 5 + x & \text{for } x < 3 \\ 4 & \text{for } x \geq 3 \end{cases}$

   g) $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} x^2 & \text{if } x \geq 2 \\ f(x) = (x^2 - 4)/(x - 2) & \text{if } x < 2 \end{cases}$

   h) $\lim_{x\to 0} f(2 + h)$ where f is a function for which $\lim_{x\to 2} f(x) = 4$

4. (a) Write out factorizations of the following:

   i. $x^2 - a^2$ ii. $x^3 - a^3$ iii. $x^3 + a^3$ iv.$x^2 + a^2$ v. $x^n - a^n$

   (b) Divide:

   i. $(3\sqrt{x} - 3\sqrt{4})/(x - 4)$ ii.$(x^5 - a^5)/(x - a)$

   (c) Calculate:

   i.$\lim_{x \to 2}(3\sqrt{x} - 3\sqrt{2})/(x - 2)$ ii.$\lim_{x \to a}(x^n - a^n)/(x - a)$

5. Graph the following functions. Use your knowledge of limits to determine the behavior near where the functions are undefined.

   (a) $(1 - |x|)/(1 + |x|)$
   (b) $\sqrt{(4 - x^2)}/(x + 2)$
   (c) $[1 - \sqrt{(1 + x^2)}]/x$

6. (a) Suppose $f(x) = 1$ when x is an irrational number and $f(x) = 0$ when x is rational. Find $f(0)$ and $\lim_{x \to 0} f(x)$.

   (b) Suppose $f(x) = 0$ when x is an irrational number and $f(x) = x^2$ when x is rational. Find $f(0)$ and $\lim_{x \to 0} f(x)$.

7. For each of the four cases below, sketch a graph of some function that satisfies the stated condition.

   (a) $\lim_{x \to 2} f(x) = 3$ and $f(2) = 4$
   (b) $\lim_{x \to 0} f(x)$ does not exist and $|f(x)| < 2$ for all x
   (c) $\lim_{x \to 0} f(x) = f(0) + 1$
   (d) $\lim_{x \to +\infty} f(x) = 1$ and $\lim_{x \to -\infty} f(x) = -1$

PDP Math 1A WS #5 Fall 1987

1. Let $f(x) = x^2$

   (a) Find the equation of the tangent lines and normal lines to the graph of $f(x)$ at $(1, 1)$ and at (a, a^2).

   (b) Find the x-intercept of the tangent line to $f(x)$ at $x = a$. Sketch $f(x)$ and the tangent line.

   (c) Find the y-intercept of the tangent line to $f(x)$ at $x = a$.

   (d) Find the y-intercept of the normal line to $f(x)$ at $x = a$.

   (e) What are the closest points on the graph of $f(x)$ to the point $(0, b)$?

   (f) Which points on the graph of f have a tangent line that pass through $(0, -9)$?

2. (a) For each of the following functions find $f'(x)$ directly from the definition, and discuss $\lim_{x\to 0} f'(x)$.
      i. $f(x) = 1/x, x \neq 0$
      ii. $f(x) = |x| - 1$
      iii. $f(x) = x^{1/3}$
      iv. $f(x) = 2x^2 + 1/x - 1$
   (b) Let $g(x) = |x^3 - 1|$. Evaluate $g'(-1), g'(0), g'(1)$.
   (c) Let $f(x) = \sqrt{2x-1}$.
      i. Express $f'(1)$ as a limit.
      ii. Find $f'(1)$ by evaluating this limit.

3. Suppose $f'(1) = A, g(1) = C$ and $g'(1) = B$.

   (a) Using the definition of the derivative directly, express $h'(1)$ in terms of $a, A, b, B,$ and C, where $h(x) = af(x) + bg(x)$.
   (b) Do the same for $k'(1)$, where $k(x) = af(x) + b/g(x)$.

4. Show that $f(x) = x^3 - 3x^2 + 3x - 6$ has exactly one real root. Between what two integers is the root?

5. A particle travels along a number line whose unit distance is one meter. At any time t seconds after a certain moment (time $t = 0$), the particle is at the point $s(t)$ on the line, where $s(t) = 4t - t^2$

   (a) What is the location of the particle at time $t = 3$?
   (b) What is its speed (ie., how fast is it moving) at time $t = 4$?
   (c) At what time(s) is the particle at rest?
   (d) What is its position when it is at rest?
   (e) For what values of t is the particle moving to the right?
   (f) For what values of t is the particle moving to the left and slowing down?

6. Match each of the derivatives shown below with the corresponding functions. (Sorry: no pictures yet)

PDP Math 1A WS#6 Fall 1987

Review

1. (a) Write the definition of $\lim_{x\to a} f(x) = L$ (without looking it up).
   (b) Sketch the derivative of each of the functions below. (Sorry: no pictures yet.)

2. A stone is thrown upward from the top of a building 50 meters high with a velocity of of 15 meters/sec. The height s of the stone at any time t is given by $s(t) = 50 + 15t - 5t^2$. Find the maximum height of the stone, how long it takes to reach this height, and the time the stone hits the ground.

   (a) Show by direct application of the definition of the derivative that

      i. $f(x) = x\sqrt{|x|}$ is differentiable at $x = 0$.

      ii. $g(x) = \sqrt{1 + |x|}$ is not differentiable at $x = 0$.

   (b) Show how the result of part 2a is reflected in the graphs of these functions.

3. From a Fall 1982 exam of Professor Ribet: Suppose ϵ is a number such that $0 < \epsilon < 1$. In terms of ϵ, find a positive number δ such that $|1/x - 1/2| < \epsilon$ is true whenever $|x - 2| < \delta$ is true.

4. This problem is from Professor Addison's Fall 1974 Math 1A midterm:

   A giddily gleeful witch, elated over passing her Mathematics 1A midterm examination, hurls a somewhat overripe pumpkin directly upward from the ground. It moves according to the law $s(t) = 96t - 16t^2$, where t is the time in seconds after it is thrown and $s(t)$ is its height in feet above the ground at time t. Find:

   (a) the velocity of the pumpkin after

   (b) 5 seconds;

   (c) the maximum height the pumpkin reaches;

   (d) the average speed of the pumpkin during its upward rise;

   (e) the acceleration of the pumpkin at its maximum height;

   (f) the rate of change of the acceleration of the pumpkin after 4 secs.

5. Find all values of p for which the following functions are (i) continuous and (ii) differentiable

   (a) $f(x) = \begin{cases} x^2 + p & x \geq 0 \\ 4 - x^2 & x < 0 \end{cases}$

   (b) $g(x) = \begin{cases} px^2 + x & x \geq 0 \\ x^3 + p^2x & x < 0 \end{cases}$

1. This problem is a practice test question. Allow yourself 15 minutes.

   (a) Find all points on the graph of $y = x^2$ whose tangent lines pass through the point $(5, 0)$.

   (b) Show that no line tangent to the graph of $f(x) = x + 1/x$ passes through the origin.

   (c) Compute $\lim_{x\to 1} f(x)$ for the following functions:

      i. $f(x) = (x^3 - 1)/(x - 1)^2$

      ii. $f(x) = (x^3 - 2x^2 + 2x - 1)/(x^3 - 1)$

2. Without looking at the book, give the definition of

   (a) "The function f is continuous at $x = a$."

   (b) "The derivative of f at $x = a$ is A."

3. (These questions are from a Prof. Lam midterm, Fall 1979.)

   (a) The graph of $y = x^3 + 3x^2 - 3x$ has two tangent lines parallel to $y = 6x + 100$. Find the equations of these two lines.

   (b) Compute $g'(1)$, where $g(x) = (x + f(x))/(x - f(x))$ and $f(1) = 4, f'(1) = 2$.

4. (a) What limit must you calculate to compute $f'(1)$, where $f(x) = x^{100}$?

   (b) Calculate

      i. $\lim_{x\to 0}[(1 + x)^{1000} - 1]/x$

      ii. $\lim_{x\to 0}[(1 + x)^{-1000} - 1]/x$

      iii. $\lim_{x\to 0}[(8 + x)^{4/3} - 16]/x$

5. (a) Convince yourself that $\lim_{h\to 0}[f(a + h) - f(h)]/h = \lim_{x\to a}[f(x) - f(a)]/(x - a)$

   (b) Use the above fact to evaluate the following limits.

      i. $\lim_{x\to 8}(x^{4/3} - 16)/(x - 8)$

      ii. $\lim_{x\to 2}[(6 + x)^{2/3} - 4]/(x - 2)$

      iii. $\lim_{x\to 1}(x^3 - 2x^2 + 2x - 1)/(x - 1)$

      iv. $\lim_{x\to -1}(x^{100} - 1)/(x + 1)$

      v. $\lim_{x\to -1}(x^2 + x)/[(x + 2)(x + 1)]$

6. At time $t = 0$ Irwin fires a rubber band at a spider on the ceiling. At time $t = 3$ it hits its mark. If $s(t)$ represents the height of the rubber band above the floor at time t , state, for each of the following pairs of quantities, which is greater.

   (a) $s(1)$ or $s(2)$

   (b) $s'(1)$ or $s'(2)$

   (c) $s''(1)$ or $s''(2)$

PDP Math 1A   WS #8   Fall 1987

1. Prove that it is impossible to find two differentiable functions f and g for which $f(0) = g(0) = 0$ and which satisfy $f(x)g(x) = x$ for all x. (*Hint: differentiate.*)

2. (a) i. If $f(x) = u(x)v(x)w(x)$, find a formula for $f'(x)$.

   ii. Find $f'(1)$, where $f(x) = (x^3 + 3x^2 - 6x + 1)(x^4 - x^2 - 1)(\sqrt{x})$.

   (b) Suppose g, u, and v are functions defined for all real numbers. Find a formula for $f'(x)$ where f is defined by

   i. $f(x) = g(u(v(x)))$

   ii. $f(x) = g(u(x) + v(x))$

   iii. $f(x) = g(u(x)v(x))$

   iv. $f(x) = g(u(x)/v(x))$

3. Suppose that $h(x) = g(u(x))$, where $u(x) = x^3 + 1$, and that $g'(1) = 2, g'(2) = 4, g'(9) = 16, g'(13) = 8, g''(1) = 5, g''(2) = 2, g''(9) = -1, and g''(13) = 12$. Find

   (a) $h'(2)$

   (b) $h''(2)$

4. A problem from Prof. Ribet's Fall 1983 Midterm

   Given that $h(x) = f(g(x))$, find the six missing values in the table below. (Sorry: no tables yet.)

5. Differentiate the following:

   (a) $f(x) = (1 + \sqrt{x})^{1/2}$

   (b) $g(x) = [(x^2 + 1)^2 + (x^2 + 1) + 1]^2$

   (c) $h(x) = (1 + (1 + (1 + x^2)^8)^8)^8$

6. Find $f'(x)$ in terms of g and g', where $g(x) > 0$ for all x. [Note: "a" is a constant, and thus so is $g(a)$.]

a) $f(x) = g(x)(x-a)$	b) $f(x) = g(a)(x-a)$
c) $f(x) = g(x+g(x))$	d) $f(x) = g(x)/(x-a)$
e) $f(x) = 1/g(x)$	f) $f(x) = g(xg(a))$
g) $f(x) = \sqrt{g(x)^2}$	h) $f(x) = \sqrt{g(x^2)}$
i) $f(2x+3) = g(x^2)$	[*Hint:* x =2$((x-3)/2)+3$]

7. Find a function $f(x)$ whose derivative at any point a is given by

$$\lim_{h \to 0} [(3a+h)^{16/5} - (3a)^{16/5}]/4h$$

Be very careful.

PDP Math 1A *WS#9* *Fall 1987*

1. (a) For which values of x are the tangent lines to the parabolas $y = x^2$ and $y = -x^2$ perpendicular?

   (b) Each point on the negative y-axis is a point of intersection of two lines tangent to the parabola $y = x^2$. At which points are these tangent lines perpendicular?

2. Let the function f be defined by $f(x) = (x^2 - 1)/(x^2 + 2|x| + 1)$.

   (a) Find the domain, range, and any asymptotes and symmetries of f.

   (b) For what values of x is f continuous?

   (c) For what values of x is f differentiable?

   (d) Sketch a graph of $y = f(x)$.

3. Calculate $\lim_{x\to 2} f(x)$ and $\lim_{x \to 0} f(x)$ for each of the following.

   (a) $f(x) = (x^3 - x - 6)/(x-2)$

   (b) $f(x) = (x^4 - x^2 - 6x)(x^2 - 2x)$

   (c) $f(x) = [(2+x)^\pi - 2^\pi]/x$

   (d) $f(x) = \begin{cases} (\sqrt{x^2+6x} - 4)/(x-2), & \text{when } x > 2 \\ 5|x|/(4x) & \text{when } x \le 2 \end{cases}$

4. Show that whatever the value of c, $x^3 - 3x + c = 0$ has at most one root in the interval $[-1, +1]$.

5. This problem is from Professor Addison's Fall 1974 Math 1A midterm:

A giddily gleeful witch, elated over passing her Mathematics 1A midterm examination, hurls a somewhat overripe pumpkin directly upward from the ground. It moves according to the law $s(t) = 96t - 16t^2$, where t is the time in seconds after it is thrown and $s(t)$ is its height in feet above the ground at time t. Find:

   (a) the velocity of the pumpkin after 1.5 seconds;
   (b) the maximum height the pumpkin reaches;
   (c) the average speed of the pumpkin during its upward rise;
   (d) the acceleration of the pumpkin at its maximum height;
   (e) the rate of change of the acceleration of the pumpkin after 4 secs.

6. Find all values of p for which the following functions are (i) continuous, (ii) differentiable

   (a) $f(x) = \begin{cases} x^2 + p & \text{when } x \geq 0 \\ 4 - x^2 & \text{when } x < 0 \end{cases}$

   (b) $g(x) = \begin{cases} px^2 + x & \text{when } x \geq 0 \\ x^3 + p^2 x & \text{when } x < 0 \end{cases}$

7. (a) Given an $\epsilon > 0$, can one always find a $\delta > 0$ such that $|1/x^2 - 1| < \epsilon$ whenever $0 < |x + 1| < \delta$?
   (b) i. Find the largest δ such that $|x^2 - 4| < .1$ whenever $|x - 2| < \delta$.
      ii. Suppose that f is a strictly increasing continuous function and that $\lim_{x\to a} f(x) = L$. Describe a procedure for finding the largest δ such that
      $0 < |f(x) - L| < 1/10$ whenever $0 < |x - a| < \delta$.
   (c) Find all values of c for which the following sentences are true.
      i. For every $\epsilon > 0$ there exists a $\delta > 0$ such that $|x^3 - 8| < \epsilon$ whenever $0 < |x - c| < \delta$.
      ii. For every $\epsilon > 0$ there exists a $\delta > 0$ such that $|x^4 - 4| < \epsilon$ whenever $0 < |x - c| < \delta$.
   (d) True or false:
      i. For every $\delta > 0$ there exists an $\epsilon > 0$ such that $|x^3 - 8| < \epsilon$ whenever $0 < |x - 2| < \delta$.
      ii. For every $\delta > 0$ there exists an $\epsilon > 0$ such that $|x^2/(1+x^2) - 2| < \epsilon$ whenever $0 < |x - 2| < \delta$.

8. This problem is from an October 1985 1A midterm:

   Assume that $f(x)$ is a differentiable function, and that the values of f and its derivative at the points $x = 0, 1, 2, 3$ are given by $f(0) = 3, f(1) = 5, f(2) = -2, f(3) = 6, f'(0) = -1, f'(1) = 0, f'(2) = 3, f'(3) = 1$ Let $g(x) = x^2 - 3x + 2$.

   (a) Calculate the derivative of $f(x)/g(x)$ at $x = 0$.

   (b) Calculate the derivative of $f(x)g(x)$ at $x = 1$.

   (c) Calculate the derivative of $f(g(x))$ at $x = 2$.

   (d) Calculate the derivative of $g(f(x))$ at $x = 3$.

PDP Math 1A WS#10 Fall 1987

1. (a) Find y' in terms of x and y for the relation $2x^2 - 3xy + 2y^2 + x - 5y + 3 = 0$.

   (b) For $x = 0$ and $x = 3$ find the associated value(s) of y and y', and use this information to give a rough sketch of the curve . How many functions $y = y(x)$ are defined implicitly by this relation?

2. (a) At which points is $g(x) = |x|$ differentiable?

   (b) Show that $g'(x) = x/|x|$.

   (c) Use the chain rule to find a formula for $\frac{d}{dx}(|f(x)|)$, and then use it to find the derivative of $h(x) = |x^2 - 4|$ at $x = 1$.

   (d) Can $\frac{d}{dx}(|f(x)|)$ exist at a point where $f(x) = 0$?

3. For each of the following relations, find y' both explicitly and implicitly, and verify that these results are in agreement. Then use the results to help sketch the graphs.

   (a) $x^{1/2} + y^{1/2} = 1$

   (b) $|x| + |y| = 1$

   (c) $x^2 + y^2 = 1$

   (d) $x^3 + y^3 = 1$

4. (a) Sketch graphs of the relations

      i. $y^2 - x^2 = 0$

      ii. $y^2 - x^2 = 1$

   (b) Find all continuous functions defined by (i) and (ii).

   (c) Compute a formula for y' for both (i) and (ii) above.

   (d) Do the functions defined by (i) and (ii) above share a common derivative?

5. (a) Sketch a graph of the relation $(y - x^2)(y - x^3) = 0$.
   (b) Find a formula for y' using implicit differentiation.
   (c) Evaluate y' at (a, a^2), where $a \neq 0, 1$.
   (d) Evaluate y' at (a, a^3), where $a \neq 0, 1$.

6. Determine $f'(x)$ at $x = -1$ for $f(x) = |(x+1)^3|$. Can you prove your answer is correct? (Be careful, $f(x)$ does have a derivative at $x = -1$.)

7. In the following problems, x and y are differentiable functions of t. Find the answers without explicitly using implicit differentiation!

   (a) If $3x + 4y = 7$ and $\frac{dx}{dt} > 0$, what is the sign of $\frac{dy}{dt}$?
   (b) If $x^2 + y^2 = 1, x(5) = \sqrt{2}/2, y(5) = -\sqrt{2}/2$ and $\frac{dx}{dt}(5) > 0$, what is the sign of $\frac{dy}{dt}(5)$?
   (c) If $x^2 + y^2 = 1, x(9) = 1$, what is the sign of $\frac{dx}{dt}(9)$?

PDP Math 1A   WS#11   Fall 1987

1. Prove or give a counter-example:

   (a) If f is continuous at 2 and assumes its maximum value at 2, then f is differentiable at 2.
   (b) If f is defined on $[0, 1]$, and continuous and differentiable on $(0, 1)$, then there exists a point x_0 in $[0, 1]$ such that $f'(x_0) = f(1) - f(0)$.

2. If f is a quadratic function of x, and if a and b are any real numbers, show that the value that satisfies the Theorem of the Mean for f is the midpoint between a and b.

3. Prove that $1/9 < \sqrt{66} - 8 < 1/8$

4. (a) Let $f(x) = |x| - 1$. Then $f(-1) = f(+1) = 0$, but $f'(x) \neq 0$ on $[1, -1]$. Does this contradict Rolle's theorem? Explain!
   (b) Does the Theorem of the Mean apply to the function $f(x) = (x^2 - 4x + 3)/(x - 3)$ on $[2, 4]$?
   (c) Is there a point c on $[2, 4]$ for which $f'(c) = [f(4) - f(2)]/(4 - 2)$, where $f(x)$ is the function of part 4b?

5. (a) Suppose that an object lies at $x = 4$ when $t = 0$ and that the velocity $dx/dt = 35$ with a possible error of ± 1, for all t in $[0, 2]$. Using the Theorem of the Mean what can you say about the object's position when $t = 2$?
   (b) The fuel consumption of an automobile varies between 17 and 23 miles per gallon, according to the conditions of driving. Let $f(x)$ be the number of gallons of fuel left in the tank after x miles have been driven. If $f(100) = 15$, find upper and lower estimates for $f(200)$.

6. Prove the following assertions:
   (a) If $f'(x) > 0$ for all x, then f is increasing.
   (b) If $f'(x) = g'(x)$ for all x, then $f(x) = g(x) + c$.
7. Suppose that f is differentiable, $f(0) = 0$ and $f(1) = 1$. Show that $f'(x_0) = 2x_0$ for some x_0 on $(0, 1)$.

PDP Math 1A *WS#12* *Fall 1987*

1. *Quadratics*
   (a) Find a quadratic polynomial which is 0 at $x = 3$, is decreasing if $x < 1$, and is increasing if $x > 1$.
   (b) Find a quadratic polynomial f which satisfies $f(0) = f'(0) = f''(0) = 2$.
   (c) Suppose that quadratic f has roots r and s. Show that $f'(r)+f'(s) = 0$.
   (d) Show that the critical point of a quadratic occurs midway between its roots.
2. *Cubics*
   (a) Find the maximum and minimum values of $f(x) = x^3 - 3x^2 + 3x + 4$ on $[0, 2]$.
   (b) Find the highest and lowest points on the graph of $f(x) = x^3 - 3x + 6$ on the following intervals: (i) $[-2, 2]$; (ii) $[-2, 3]$; and (iii) $(-2, 3)$.
   (c) Sketch a graph of $f(x) = x^3 - 3x + 6$ indicating local maxima, minima, and points of inflection.
   (d) Find the equation of the cubic for which the origin is a point of inflection, and $(-2, 16)$ are the coordinates of the local maximum point.
   (e) Show that the maximum and minimum values of the function $f(x) = x^3 + ax^2 + bx + c$ on the interval $[p, q]$ occur at the endpoints if $a^2 < 3b$.
3. (a) Suppose that $f'(x) < 0$ for all real numbers x. What can you say about the minimum and maximum values of f on the interval $[a, b]$?
   (b) Find the coordinates of the point or points on the graph of $x^2 + y^2 = 1$ closest to $(2, 0)$.
4. Given the function $f(x) = x^n$, where n is a positive integer, show that f has a minimum at $x = 0$ if n is an even integer and that f has a point of inflection at $x = 0$ if n is an odd integer.

5. Determine asymptotes, symmetry, intercepts, relative maxima and minima, inflection points, regions where the graphs are increasing or decreasing, and discuss concavity. Sketch each graph. Each has appeared on a Math 1A midterm.

   (a) $f(x) = x^4 + 4x$
   (b) $f(x) = x/(x^2 + 4)$
   (c) $f(x) = x^2/(x^2 - 4)$
   (d) $f(x) = [(1 + x)\sqrt{1 - x^2}]/x$

6. Show that $|x + (1/x)| \geq 2$ for all $x \neq 0$. *Hint: Find the maxima and minima of a certain function.*

7. (a) Show that $f(x) = x^5 + 10x^3 + 45x - 10$ has exactly one real root. Find the integer closest to it.
   (b) Show that $f(x) = x^5 - 10x^3 + 45x - 10$ has exactly one real root. Find the integer closest to it.
   (c) Find a seventh degree polynomial that has exactly one real root.

8. Show that the function

$$f(x) = (ax + b)/(cx + d)$$

has no relative maxima or minima, where a, b, c, d are any numbers such that $ad - bc \neq 0$.

PDP Math 1A　　WS#13　　Fall 1987

Note: Worksheet #14 will be on max-min word problems, section 6.6. Be sure to come to workshop having studied this material thoroughly.

1. Use the appropriate linear approximation to estimate the following:

   a) $\sqrt{16.03}$　　b) $\sqrt{15.97}$　　c) $\sqrt{16.06}$　　d) $\sqrt{15.94}$

   What's going on here?

2. Suppose, in each of the cases below, I were to estimate $f(a+h)$ using the linear approximation to f at $x = a$, i.e., $f(a + h) = f(a) + f'(a)h$. When would I get an underestimate? an overestimate?

a) $f(x) = x^2$	$a = 1$	$h = .1$	*b)* $f(x) = 5x - 23$	$a = 2$	$h = .23$
c) $f(x) = x^3$	$a = 0$	$h = .1$	*d)* $f(x) = x^3$	$a = 0$	$h = -.1$
e) $f(x) = x^{1/2}$	$a = 1$	$h = 1$	*f)* $f(x) = x^{1/2}$	$a = 1$	$h = -1.3$
g) $f(x) = x^{1/3}$	$a = 1$	$h = -9$	*h)* $f(x) = x^{1/3}$	$a = 1$	$h = -10$

3. Suppose $a < b$, and I know the exact value of both $\sqrt{a}$ and $\sqrt{b}$. To estimate the square root of $(a + b)/2$ should I use the linear approximation to $\sqrt{x}$ at $x = a$ or at $x = b$? Reasons please!

4. A coat of paint of thickness t cm is applied evenly to the faces of a cube of edge b cm. Use differentials to find approproximately the number of cubic centimeters of paint used. Compare this with the exact amount used by computing volumes before and after painting.

5. (a) Estimate $1/1.01$ by examining $f(x) = 1/(1 + x)$ near $x = 0$. Is your estimate greater than or less than the true value?

   (b) A physics textbook asserts that for $|x|$ small, $1/(1+x) = 1-x$. What is meant by this statement?

   (c) If you travel 1 mile in $60 + x$ seconds, what is your speed in m.p.h.?

   (d) Show that a good approximation to the speed in part (5c) is $(60 - x)$ m.p.h.

6. A ladder, 10 feet long, is leaning flush against a wall. The bottom of the ladder is pulled away from the wall at the constant rate of 1/2 foot per second.

   (a) How far does the top of the ladder fall during the first four seconds of motion? The next four? The next four? The next four? The last four? *Use your calculators; find answers to two decimal places.*

   (b) Based on the evidence in (6a), what can you say about the rate at which the top of the ladder is falling?

   (c) How fast is the top of the ladder approaching the ground when the bottom of the ladder is 6 feet from the wall?

   (d) How fast is the length of the ladder changing when the bottom of the ladder is 6 feet from the wall?

   (e) How fast is the top of the ladder moving when the ladder hits the ground? What is the physical significance of your answer?

   (f) The ladder, the wall, and the ground form a triangle. How fast is the area of this triangle changing when the ladder is 6 feet from the ground? Is the triangle getting larger or smaller?

   (g) How fast is the distance between the wall and the midpoint of the ladder changing when the ladder is 6 feet from the wall?

   (h) How fast is the distance between the midpoint of the ladder and the base of the wall changing when the ladder is 6 feet from the wall? If you struggled with this, does the answer suggest an easier way to do this problem?

PDP Math 1A WS#14 Fall 1987

Worksheet #15 will be on related-rates word problems, section 6.9. You should be familiar with this material before coming to workshop.

1. (a) Find the dimensions of the rectangle of largest area that can be inscribed in the region bounded by the x-axis and the graph of $y = 1 - x^2$.

   (b) Find the dimensions of the rectangle of largest area that can be inscribed in the triangle bounded by the x-axis, the y-axis, and the line $(x/a) + (y/b) = 1$.

2. Of all the triangles that pass through the point (2,1) and have two sides on the coordinate axes, find the dimensions of the one having the smallest area.

3. The strength of a beam with rectangular cross section is proportional to the product of its width w and the square of its height h. Find the dimensions of the strongest beam that can be cut from a cylindrical log having a circular cross section with radius 12 inches.

4. What are the dimensions of the lightest cylindrical aluminum can with capacity $1000cm^3$?

5. (a) A rectangular poster is to have side margins of $2''$, top and bottom margins of $4''$, and 50 sq. in. of printed matter. What are the dimensions of the poster of least area that meets these specifications?

   (b) A sheet of paper contains 18 square feet. The top and bottom margins are $9''$ and the side margins are $6''$. What are the dimensions of the page that has the largest printed area?

6. A man has 100 meters of fencing and wishes to use it to enclose three equal and contiguous pens, using an existing building as a common side of the three pens. What are the dimensions of the pen arrangement that gives the most total area inside the pens?

7. (a) Suppose that $f'(x) < 0$ for all real numbers x. What can you say about the minimum and maximum values of f on the interval $[a, b]$?

   (b) An island is at point A, 6 miles off shore from the nearest point, B, on a straight beach. A store is located at point C which is 7 miles from B along the shore. If a woman can row at the rate of 4 m.p.h. and can run at the rate of 5 m.p.h., to what point on the shore should she row in order to get to the store as fast as possible? (Sorry: no pictures yet.)

8. (a) Two hallways $8'$ and $27'$ wide meet at a right angle. What is the length of the longest ladder that can be carried horizontally down one hallway into the other?

   (b) Find the shortest line segment with endpoints on the positive x- and y- axes that passes through the point (1,8).

   (c) The sidewall of a building is to be braced by a beam which must pass over a parallel wall $10'$ high and $8'$ from the building. Find the length of the shortest beam that can be used.

PDP Math 1A WS#15 Fall 1987

1. Given that a spherical raindrop evaporates at a rate proportional to its surface area, how fast does the radius shrink?

2. Two airplanes are flying north at the same height on parallel paths ten miles apart with speeds of 400 and 600 miles per hour. How fast is the distance between the planes changing when the slow plane is five miles further north than the fast one?

3. Sand is flowing from a pipe at the rate of s cubic meters per second, and falling in a conical pile. The diameter of the base of this conical pile is always 3 times the altitude. At what rate is the altitude of the pile increasing when the altitude is h meters?

4. A spherical weather balloon is being inflated at a rate of $.5m^3/sec.$

   (a) How fast is the diameter increasing at the instant the diameter is 2 meters?

   (b) How fast is the volume changing at that same instant?

   (c) How fast is the surface area changing at that same instant?

5. (a) An arc light is 5 meters above a sidewalk. A man 2 meters tall walks away from the point under the light at a rate of 2 meters/second. How fast is his shadow lengthening when he is 7 meters away from the point under the light?

   (b) A light is on the ground 40 m away from a building. A man 2 m tall walks from the light toward the building at 2 m/sec. How rapidly is his shadow on the building growing shorter when he is 20 m from the building?

   (c) A light is at the top of a pole which is h meters high. A ball is dropped from half the height of h at a point which is at a horizontal distance d meters from the pole. Assume that the ball falls according to the law $s = gt^2$, where t is the time in seconds, s is the distance in meters, and g is a constant. Find how fast the tip of the shadow of the ball is moving along the ground t_0 seconds after it is dropped.

6. $BDCE$ is a rectangle with BC of length 1 in. and BE of length 2 in. Q moves from D to E at a uniform rate of 1 in./min. starting from D at time $t = 0$; and P moves from C toward D at the same rate, starting from C at time $t = 0$.

   (a) During the one-minute period beginning at time $t = 0$ what is the minimum area of the triangle BPQ, and what is its maximum area? *Hint: there are three right triangles in the picture.*

   (b) At one-half minute after time $t = 0$, is the length of PQ increasing or decreasing, and how fast?
   (Sorry, no pictures yet.)

7. (a) Rain is falling at the rate of q inches per hour into an open conical tank of height H and radius R. Show that at each instant the rate at which water is rising in the tank is

$$q \times \frac{\text{area of tank opening}}{\text{area of water surface}}$$

   (b) Show that the result of (7a) is true for an open tank of arbitrary shape.

PDP Math 1A *WS#16* *Fall 1987*

1. Evaluate the following:

   (a) $\sum_{i=1}^{100} 2$
   (b) $\sum_{i=1}^{100} 1/(i+3) - 1/(i+4)$
   (c) $\sum_{i=1}^{\infty} 1/(i(i+1))$ *Hint:*$(1/i) - (1/(i+1)) = ?$
   (d) $1/6 + 1/12 + 1/20 + 1/30 + 1/42 + \ldots$

2. (a) By choosing a suitable partition of [1,2] , show that $\int_1^2 (1/t)dt$ lies between $(1/4)(4/5+2/3+4/7+1/2)$ and $(1/4)(1+4/5+2/3+4/7)$.

   (b) Suppose that to evaluate $\int_a^b f(t)dt$ you partition $[a, b]$ and average the upper and lower sums for f . If $f''(x) < 0$ on $[a, b]$, what can you say about your estimate?

3. Using high-school geometry, compute the following:

a) $\displaystyle\int_0^2 x\,dx$ b) $\displaystyle\int_1^4 (1+3x)\,dx$

c) $\displaystyle\int_{-1}^3 (1-x)\,dx$ d) $\displaystyle\int_0^1 (1-x^2)^{1/2}\,dx$

e) $\displaystyle\int_0^{1/2} (1-x^2)^{1/2}\,dx$

4. A car travelling at $v_0\ m/sec$ applies its brakes at time $t = 0$. The brakes furnish a negative acceleration of $-b\ m/sec^2$, where $b > 0$.

   (a) What is the velocity of the car as a function of time for $t > 0$?

   (b) What is the distance the car travels as a function of time for $t > 0$?

   (c) How much time elapses before the car stops?

   (d) How far does it travel before it stops?

5. Let $f(x) = cx + d$ on $[a, b]$, where $a, b, c, d > 0$, and suppose there is a partition P of $[a, b]$ into n equal subdivisions.

   (a) Find $A = \int_a^b f(x)dx$ by interpreting the definite integral as an area.

   (b) Exactly how much smaller is A than the upper sum for f on P?

   (c) Exactly how much larger is A than the lower sum for f on P?

   (d) Where in each subinterval should ξ_i be chosen so that the area A is exactly equal to the approximation to the area given by the sum

$$\sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

6. (a) For $0 \le x \le 1$, let $f(x) =$ the first digit in the decimal expansion of x; e.g., $f(.713) = 7, f(1/4) = 2$.

      i. Draw a graph of f.

      ii. Compute $\int_0^1 f(x)dx$

   (b) Let $g(x) = [x] =$ "the greatest integer" in x (the largest integer smaller than or equal to x); e.g., $[5/2] = 2, [.4] = 0,$ and $[-1.2] = -2$.

      i. Draw a graph of g .

      ii. Compute $\int_{-100}^{100} g(x)dx$

7. Let $f(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$

   and consider a subdivision Δ of $[0, 1]$ of one hundred equal subintervals: $0 < .01 < .02 < .03 < ... < .99 < 1$

   (a) Compute the lower sum $LS(\Delta)$ of the function f on $[0, 1]$.

   (b) Compute the upper sum $US(\Delta)$ of the function f on $[0, 1]$.

   (c) Is f integrable on $[0, 1]$? Give reasons for your answer.

PDP Math 1A WS#17 Fall 1987

1. Suppose $\int_{-a}^{a} f(x)dx = c\int_0^a f(x)dx$

   (a) If f is an odd function, what must the value of c be?

   (b) If f is an even function, what must the value of c be?

2. Evaluate the following:

   (a) $\int_0^1 (x+y)dx$

   (b) $\int_0^1 (x+y)dz$

   (c) $\int_0^1 (1+y)dx$

   (d) $\int_0^1 (\int_0^y (1+t)dx)dy$

   (e) $d/dt[\int_0^1 (\int_0^y (1+t)dx)dy]$

3. A bus moves along the x-axis with velocity $v(t) = t^2 - 4t + 3$. First express each of these quantities as an integral, and then find its value:

   (a) the displacement of the bus between $t = 0$ and $t = 5$.

   (b) the actual distance the bus travels during this time.

4. Write upper and lower sums for the integrals of the following functions on [0,1]. Assume that [0,1] has been divided into 100 equal subdivisions.

   (a) $f(x) = x^2$

   (b) $f(x) = 1 - x^{1/2}$

5. The density (weight per unit length) of a straight wire of length L varies along its length according to the formula $\delta(x) = kx + b\ g/cm\ (0 < x < L)$ where k and b are positive constants.

   (a) Draw a graph of $\delta(x)$ and interpret the weight of the wire in terms of some feature of this graph.

   (b) Write a definite integral that gives the weight of the wire.

   (c) Evaluate this integral using either geometry or calculus.

6. Suppose the interval [a,b] is partitioned into n subdivisions of equal length, and consider upper and lower sums for the function $f(x)$ on [a,b].

   (a) If $f(a) > 0$ and $f'(x) > 0$ on [a,b] , find the coordinates of the four corners of the i^{th} rectangle of the upper sum for f on [a,b].

   (b) Do the same for the lower sum.

PDP Math 1A WS#18 Fall 1987

1. Show that $-3 < \int_{-1}^{2}(x^2-1)/(x^2+1)dx < 2$. *Hint: Find the minimum and maximum values of the integrand on [-1,2].*

2. Evaluate the following integrals

   (a) $\int_1^3 x/\sqrt{x^2+5}\, dx$ *Hint: Let* $f(x)=\sqrt{x^2+5}$. *What is* $f'(x)$*?*

   (b) $\int_1^2 (2x+1)/(x^2+x-1)^2\, dx$

   (c) $\int_1^3 (x-3/x)^5(1+3/x^2)\, dx$

   (d) $\int_{-1}^{3} (x-3/x)^5(1+3/x^2)\, dx$

3. Calculate the following derivatives using the chain rule and the fundamental theorem of calculus.

   (a) $\frac{d}{dx}\int_0^x \frac{dt}{1+t^2}$

   (b) $\frac{d}{dx}\int_0^{x^2} \frac{dt}{1+t^2}$

   (c) $\frac{d}{dx}\int_{-x^2}^{x^2} \frac{dt}{1+t^2}$

   (d) $\frac{d^2}{dx^2}\int_0^x \frac{dt}{1+t^2}$

4. A point is moving along the x-axis: At $t=0$ it is at the origin; for $0 \le t < 1$ its velocity is given by $v(t)=2t-1$; for $1 \le t < 2$ its velocity is given by $v(t)=4t-2$, for $2 \le t \le 3$ its velocity is given by $v(t)=6t-3$. Where is the point at $t=3$? Sketch a rough graph of the point's motion.

5. Suppose $f(x)=1/[2\sqrt{x+1}]$. Evaluate the following definite integrals:

   (a) $\int_0^3 f'(x)\, dx$

   (b) $\int_0^3 f(x)\, dx$

   (c) $\int_0^3 [\int_0^x f'(t)\, dt]\, dx$

6. Let f be an integrable function and define

$$F(x)=\int_c^x f(t)\, dt \qquad G(x)=\int_d^x f(t)\, dt$$

   Show that F and G differ by a constant, and find what that constant is. *(Hint: this problem is not hard. Draw a graph to see what* F *and* G *look like. Then compute* $F(x)-G(x)$ *.)*

1. Evaluate the following integrals:

   (a) $\int x(1-x^2)^{-99/2}\,dx$

   (b) $\int x^{-1/2}(1-x^2)\,dx$

   (c) $\int (1-x)^{-4/3}\,dx$

   (d) $\int (1-2x+x^2)^{2/3}\,dx$

   (e) $\int (2x+3)(x^2+3x+2)^{-2}\,dx$

   (f) $\int (x)^{-1/2}(1+x^{1/2})^{-3}\,dx$

2. Sketch and set up integrals for the area of the regions bounded by the given curves. DO NOT EVALUATE THE INTEGRALS

   (a) $y = x^2$ and $y = 12 - x$.

   (b) $y = x^2, y = -1, x = -1$ and $x = 1$.

   (c) $y = x^2, y = x + 2, x = 0$, and $x = 2$.

   (d) $y = x^3$ and $y = x$.

   (e) $y = x^3$ and $y = 3x + 2$.

3. Evaluate the following definite integrals. Could you evaluate any of them as indefinite integrals? Why or why not?

   (a) $\int_2^3 (t^3-1)/(t-1)\,dt$

   (b) $\int_0^3 |t-1|\,dt$

   (c) $\int_0^1 (1-x^2)^{1/2}\,dx$

   (d) $\int_0^1 x^{-1/3}(1-x^{4/3})^{1/2}\,dx$

   (e) $\int_{-1}^1 (x^{17}+x^{31})/(1+x^2+x^4)\,dx$

4. (a) The region under the graph of $y = -2x + 4$ on [-2,1] is to be divided into two parts of equal area by a vertical line. Where should the line be drawn?

   (b) Where would you draw a horizontal line to divide the region in part (4a) two parts of equal area?

5. (a) At time $t = 0$, a container holds 1 liter of water. Water is poured into the container at the rate of $3t^2 - 2t + 3$ liters per minute (t equals time in minutes.) If the container has a leak which drains 2 liters per minute, how much water is in the tank at the end of three minutes?

   (b) What if four liters per minute leaks from the tank?

   (c) What if eight liters per minute leaks from the tank? *[Hint: What happens if the tank is empty for a while?]*

PDP Math 1A   WS#20   Fall 1987

1. Find the following antiderivatives.

   (a) $\int x(1-x^2)^{1/2}dx$

   (b) $\int x^2(1-x)^{1/2}dx$

   (c) $\int x^3(1-x^2)^{1/2}dx$

2. We are given a differentiable, odd function f defined on [-3, 3] which has zeros at $x = -2, 0, 2$ (and nowhere else) and critical points at $x = -1, 1$ (and nowhere else). Also we know that $f(-1) = 1$. Define a new function F on [-3, 3] by the formula

$$F(x) = \int_{-2}^{x} f(t)dt$$

   (a) Sketch a rough graph of f .

   (b) Find the value of $F(-2), F(2)$, and an upper and lower bound on $F(0)$.

   (c) Find the critical points and inflection points of F on [-3,3] .

   (d) Sketch a rough graph of F on [-3,3] .

   (e) Interpret the points found in 2c in terms of the graphs of both f and F .

3. (a) If $\int_a^b f(x)\,dx = b^3 - a^3$ for all numbers a and b , what is $\int_a^b f'(x)\,dx$?

   (b) If $\frac{d}{dx}(\int_a^x f(t)\,dt) = x^3 - 1$, what is $\int_a^b f'(x)\,dx$?

   (c) If $\int_a^b f(u(x))u'(x)\,dx = (2/3)(b^2+1)^{3/2} - (2/3)(a^2+1)^{3/2}$ for all numbers a and b , what might $f(x)$ and $u(x)$ be? Are they unique?

4. (a) Consider the points on the graph of $y = x^2$ whose y coordinates are y_0 . How far are these points from the y-axis?
   (b) Find the area bounded by $y = x^2$ and the line $y = 1$ by integrating along the y-axis.
   (c) Find the area of the triangle formed by the line $y = -2x + 4$ and the x- and y- axes in two ways: first integrate along the x-axis and then along the y-axis .
   (d) Find the area bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$ in two ways: first integrate along the x-axis and then along the y-axis. Which method is easier?

5. Find formulas for the values of these definite integrals, where a, b, c, d are constants. Do any restrictions apply?
   (a) $\int_a^b (cx + d)^n \, dx$
   (b) $\int_a^b x(cx^2 + d)^n \, dx$

6. Consider the unit circle sketched below. Write as a definite integral (or sum or difference of integrals) the areas of the regions listed to the right of the figure. DO NOT EVALUATE THE INTEGRALS

   (Sorry: no pictures yet!)

PDP Math 1A *WS#21* *Fall 1987*

1. (a) Show that $\sin x < x$ for $0 < x < \pi/2$. *[Hint: Mean Value Theorem]*
   (b) Show that $\tan x > x$ for $0 < x < \pi/2$.
   (c) Show that for small x, $\sin x \neq x \neq \tan x$.
   (d) Graph the functions x , $\sin x$, and $\tan x$ on $[0, \pi/6]$.

2. Evaluate
   (a) $\int \sin^2 x \, dx$
   (b) $\int \cos^2 x \, dx$
   (c) $\int \sec^2 x \, dx$
   (d) $\int \tan^2 x \, dx$
   (e) $\int \sin^5 x \, dx$
   (f) $\int \sin^5 x \cos^5 x \, dx$

3. Graph the following functions, showing zeros, maxima, minima, and inflection points.
   (a) $f(x) = \sin x + \cos x$
   (b) $f(x) = \sin x \cos x$
   (c) $f(x) = \sin^2 x$
   (d) $f(x) = \sqrt{\sin^2 x}$
4. (a) Evaluate $\int_0^{2\pi} (1 - \cos^2 x)^{1/2}\, dx$ *(Hint: We hope you didn't get zero.)*
   (b) Show that for any integer n, $\int_0^{\pi/2} \cos^2 nx\, dx$ is independent of n .
   (c) Let $f(x) = \int_0^x |\cos t| dt$, $\pi/2 \le x \le 3\pi/2$. Write f without using integrals.
5. Find an equation of a line through the point (-3,2) making an angle of $\pi/4$ with the line $3x - 2y - 7 = 0$.
6. Find the area between $f(x) = \sin x$ and $g(x) = \sin^3 x$ on $[0, p]$.
7. (a) Find the equation of the tangent line to $\tan x + \cot y = 2$ at $(\pi/4, \pi/4)$.
   (b) Also try the equation $(2x \sin y)^2 - x + 2 \sin y = 1$ at $(1, \pi/6)$.
8. (a) By inspection, find a solution of the equation $\tan x = x$.
   (b) Using your calculators, find to three decimal places a number x_o on $(\pi, 3\pi/2)$ which is also a solution to $\tan x = x$.
   (c) Define the function $f(x) = \sin x/x$ at $x = 0$ so as to make f continuous, then sketch the graph of f on $[-2\pi, 2\pi]$.

PDP Math 1A   WS#22 (Sample Second Midterm)   Fall 1987

1. Evaluate the antiderivatives.
   (a) $\int x(x^2 + 1)^{-1/2}\, dx$
   (b) $\int (2x^2 + 4x - 1)^{1/2}(x + 1)\, dx$
   (c) $\int \cos^3 x\, dx$
   (d) $\int x \sin^3(x^2)\, dx$
2. Maximize the area of a rectangle R inscribed in a right triangle whose legs have lengths a and b . (You may assume that the sides of the rectangle lie along the legs of the triangle.)
3. Using differentials, give approximations for these quantities. (Show your work.)
   (a) $1/(\sqrt{10} + 1)$
   (b) $3 + 4\tan(\pi/12)$

4. Sketch the bounded region formed by the curves $y = x^4 - 2x^2$ and $y = 2x^2$, and find its area.

5. Let $f(x) = x^2/2 + 1/x$. Determine the following characteristics of f.

   (a) zeros
   (b) where $f(x)$ becomes infinite
   (c) local maxima and minima
   (d) where $f(x)$ is increasing or decreasing
   (e) intervals of concavity or convexity
   (f) a sketch of the graph of $f(x)$

6. Let a and b be positive constants. Prove that if x is positive then

$$a^3x^2 + b^3/x \geq (2^{1/3} + 2^{-2/3})ab^2$$

7. The area of a square is changing at the rate of $k\ cm^2/sec$. How fast is the perimeter changing when the area is $A\ cm^2$?

8. A particle moves along a straight line in such a way that at any time t its acceleration is $a(t) = 6t^2 + 2t$. Find the distance the particle has traveled by time t_0 , given that the position at $t = 0$ is s_0 and the velocity at $t = 0$ is v_0 .

9. At a certain moment, ship A is 6 miles south and 8 miles west of ship B . Ship A at that moment is steaming east at 12 mph, while ship B is steaming north at 15 mph. Are the ships approaching each other or separating from each other? At what rate?

10. Suppose that $f(x)$ has the following definition:

$$f(x) = \int_0^x (1 + \sin^2 t)^{1/2} dt$$

   (a) What is $f''(x)$?
   (b) Express $g(x) = \int_2^{\cos x} (1 + \sin^2 t)^{1/2} dt$ in terms of f . Then find $g'(x)$.

11. Find the tangent line to the curve $f(x) = 2\sin x + 3\cos x + 5\tan x$ at $x = \pi/6$.

12. The math department needs a substitute professor to give one 12,000 word lecture on calculus. They don't care how long it takes the professor to deliver the lecture, but they do care how much it costs. The different professors available speak at different rates of speed, and professors who speaks at the rate of v words/minute charge $16 + 10^{-6}v^3$ dollars per hour for their services. How should v be chosen so as to minimize the fee paid for the lecture?

1. Evaluate these antiderivatives.
   (a) $\int 1/(7+3x^2)\ dx$
   (b) $\int x/(1-4x^4)^{1/2}\ dx$
   (c) $\int 1/[x(x^3-1)^{1/2}]\ dx$ *(Hint: let* $u^2 = x^3 - 1$*)*
2. (a) Let $f(x) = \arcsin x + \arccos x$. Give two separate arguments to show that f is a constant. What is that constant?
   (b) Let $f(x) = \arcsin(\cos x)$, $0 \le x \le \pi$. Show that $f(x) = ax + b$ for constants a and b , and find the values of these constants.
   (c) Let $f(x) = \arctan x$. Prove that $f'(x) = 1/(1+x^2)$.
3. Use differentials to approximate these quantities.
   (a) $\arctan(1.04)$.
   (b) $f(\pi/12)$, where $f(x) = \int_0^{\sin x} 1/\sqrt{1-t}dt$.
4. (a) On the interval $-2\pi \le x \le 2\pi$ superimpose a graph of $y = (-2/3\pi)x$ on a graph of $y = \sin x$, and approximately locate any intersections.
   (b) Find a positive value of m such that $y = -mx$ intersects $y = \sin x$ exactly three times. *[Hint: a solution to the equation* $\tan x = x$ *is* $x \cong 4.4934$ *radians.]*
5. Suppose the function g is the inverse of the function f. Show the plausibility of the following statements in terms of properties of the graphs, and then give a one line proof of each.
   (a) If f is decreasing at a point, then g is also decreasing at the point.
   (b) If f is decreasing at a point, then f and g are either both concave up or both concave down at the point.
   (c) If f is increasing at a point, then g is also increasing at the point.
   (d) If f is increasing at a point, then f and g have opposite concavity at the point.
6. Solve for x:
   (a) $x^{1/10} = 1000$
   (b) $3\log_x 4 = 2$
   (c) $3^x = 4^{2x-1}$
   (d) $\log_x(1-x) = 2$
   (e) $\log_2 x = \log_4 5 + 3\log_2 3$

7. Suppose that g and h are increasing functions on an interval I. For the following functions, either show that they must be increasing on I, or give a counterexample.

   (a) $g + h$

   (b) $g \cdot h$

   (c) $g \circ h$

PDP Math 1A WS#24 Fall 1987

1. Differentiate

   (a) $f(x) = x^{\cos x}$

   (b) $f(x) = (\cos x)^x$

   (c) $f(x) = 10^{(e^{(x^2)})}$

   (d) $f(x) = (x^x)^x$

   (e) $f(x) = x^{(x^x)}$

2. Integrate

   (a) $\int xe^{(x^2)}\,dx$

   (b) $\int \cos x e^{\sin x}\,dx$

   (c) $\int \sqrt{10^{3x}}\,dx$

   (d) $\int f'(x)e^{f(x)}\,dx$

3. (a) For what values of b is $f(x) = \log_b x$ an increasing function?

   (b) Sketch the functions $\log_e x$ and $\log_{1/e} x$.

   (c) For what values of b is $f(x) = b^x$ an increasing function?

   (d) Sketch the functions e^x and $(1/e)^x$.

4. Evaluate the following integrals

   (a) $\int x/(x^2 + 1)\,dx$

   (b) $\int x/(x + 1)\,dx$

   (c) $\int 1/(x \ln x)\,dx$

   (d) $\int \tan x\,dx$

   (e) $\int f'(x)/f(x)\,dx$

   (f) $\int 1/(1 + e^x)\,dx$ *(Hint: Divide)*

5. For each of the following functions: (i) Analyze the zeros and the relative maxima and minima; (ii) Find the equation of the line tangent to the graph that passes through the origin. (iii) Sketch.

   (a) $f(x) = \ln x/x,\ \textit{for } x > 0$
   (b) $f(x) = e^x/x$
   (c) $f(x) = x^{\ln x},\ \textit{for } x > 0$
   (d) $f(x) = x^x$

6. Find the domain and range of the following functions:

   (a) $f(x) = e^{\sin x}$
   (b) $f(x) = \ln(\sin x)$
   (c) $f(x) = \sin(e^x)$
   (d) $f(x) = \sin(\ln x)$
   (e) $f(x) = \exp(x^2 + x + 1)$
   (f) $f(x) = \ln(x^2 + x - 2)$

7. (a) Show that $(\log_e \pi)(\log_\pi e) = 1$. Generalize.
   (b) By analyzing the function $f(x) = \ln x/x$, show that $\pi^e < e^\pi$.
   Generalize.

PDP Math 1A WS#25 Fall 1987

1. Give an equation in terms of x and y and sketch a graph.

   (a) $x = 5\cos t, y = 3\sin t$
   (b) $x = 5\sec t, y = 3\tan t$
   (c) $x = \cos^3 t, y = \sin^3 t$
   (d) $x = \log_{10} t, y = \ln t$
   (e) $x = 10^t, y = e^t$
   (f) $x = t^m, y = t^n$

2. For each of the following sets of parametric equations: (i) Graph in the x-y plane. (Caution: they are all different graphs.) (ii) What do these graphs have in common? (iii) Show how each graph is traced as values of t go from $-\infty$ to 0 to $+\infty$.

   (a) $x = t, y = t^2$
   (b) $x = t^2, y = t^4$
   (c) $x = e^t, y = e^{2t}$
   (d) $x = (1 - t^2)^{1/2}, y = 1 - t^2$
   (e) $x = t^{-1}, y = t^{-2}$
   (f) $x = |t|^{1/2}, y = t$

3. A bullet shot with initial velocity v_o from a gun at ground level aimed up at an angle θ travels according to the parametric equations $x(t) = v_o t \cos\theta y(t) = v_o t \sin\theta - gt^2$.

   (a) Show that the path of the bullet is a parabola.
   (b) How much time elapses before the bullet hits the ground?
   (c) How far does the bullet travel before it hits the ground?
   (d) What is the maximum height reached?
   (e) How should θ be chosen to maximize the range?

4. Consider the parametric equations $x(t) = e^t + e^{-t}, y(t) = e^t - e^{-t}$.

   (a) Eliminate the parameter t and find an equation in terms of x and y .
   (b) Graph the function in the x-y plane, being careful to consider domain and range.
   (c) Show that if t represents time, then the speed of the point $(x(t), y(t))$ is numerically equal to the distance of the point from the origin.

5. (a) Sketch, identify by name, and find the arclength of the curve

$$x(t) = \begin{cases} t & \text{for } 0 \le t \le 1 \\ 1 & \text{for } 1 \le t \le 2 \\ 3 - t & \text{for } 2 \le t \le 3 \\ 0 & \text{for } 3 \le t \le 4 \end{cases} \qquad y(t) = \begin{cases} 0 & \text{for } 0 \le t \le 1 \\ t - 1 & \text{for } 1 \le t \le 2 \\ 1 & \text{for } 2 < t < 3 \\ 4 - t & \text{for } 3 \le t \le 4 \end{cases}$$

   (b) Find parametric equations for this curve with reversed orientation.

6. Given any point (x,y) , let a parameter of the point be the angle θ from the positive x-axis to a line connecting the origin to the point. Find equations in terms of the parameter q for the following curves:

   (a) the circle $x^2 + y^2 = R^2$

   (b) the parabola $y = x^2$

   (c) the line $x + y = 1 (x, y > 0)$

   (d) the curve $y = xe^x$

PDP Math 1A   WS#26   Fall 1987

There will be no Wednesday-Thursday meetings of the workshop this week. Happy Thanksgiving! However, take note. Experience has shown that the Thanksgiving break is a crucial period for Math 1A students, and those who leave their review for the final until after the break find themselves hopelessly far behind. Moreover, there is a whole chapter (chapter 14 on methods of integration) that has not yet been covered, and there is only one workshop that will deal with this material (Monday-Tuesday, Nov.30-Dec.1). See the back of this worksheet for a schedule of post-break workshops and review sessions.

1. For the following sets of parametric equations, sketch the curve in the x-y plane, and find for what values of t (if any) the slope is $0, \pm 1,$ or ∞ .

   (a) $x = e^{\cos t}, y = e^{\sin t}$

   (b) $x = t^3, y = t^2$

   (c) $x = t^2, y = t^3 - t$

2. Find the lengths of the following curves over the given intervals. Sketch.

   (a) $x = t^2, y = t^3$ for $0 \le t \le 1$

   (b) $x = \cos^3 t, y = \sin^3 t$ for $0 \le t \le \pi/2$

   (c) $y = 1/2(e^x + e^{-x})$ for $0 < a \le x \le b$

3. For the following curves, i) Find the curvature as a function of x. ii) Find the maximum values of the curvature. iii) Find the minimum values of the curvature. iv) Sketch the circle of curvature at the points of ii).

   (a) $y = x^2$

   (b) $y = \sin x$

   (c) $y = e^x$

4. Let C be the curve given in parametric equations for $0 < t \leq 1$ by $x(t) = t\cos(2\pi \ln t), y(t) = t\sin(2\pi \ln t)$.

   (a) How far from the origin is $(x(t_0), y(t_0))$?
   (b) Sketch the curve from $t = 1$ to $t = e^{-1}$ and find its length. (Use $t = 1, e^{-1/4}, e^{-1/2}, e^{-3/4}, e^{-1}$ in doing the sketch.)
   (c) Sketch the curve from $t = e^{-1} tot = e^{-2}$ and find its length.
   (d) Sketch the whole curve C and find its length.

5. A bicycle moving with velocity v has a rear wheel of radius R with a reflector at a distance r from the center.

   (a) What are the maximum and minimum vertical components of the reflector's velocity?
   (b) What are the maximum and minimum horizontal components of the reflector's velocity?
   (c) Is the horizontal or vertical component of the velocity ever zero?
   (d) Repeat the problem for the special case of $r = R$.

PDP Math 1A WS#27 Fall 1987

1. These integrals are useful in trigonometric substitution. You should remember them or be able to evaluate them quickly.

   (a) $\int \sin^2 \theta \, d\theta$
   (b) $\int \cos^2 \theta \, d\theta$
   (c) $\int \sec^2 \theta \, d\theta$
   (d) $\int \tan^2 \theta \, d\theta$
   (e) $\int \sec \theta \, d\theta$
   (f) $\int \csc \theta \, d\theta$
   (g) $\int \tan \theta \, d\theta$
   (h) $\int \cot \theta \, d\theta$

2. Evaluate.

   (a) $\int (9x^2 - 4)^{1/2} x^{-1} \, dx$
   (b) $\int (9 - x^2)^{-1/2} x^{-2} \, dx$
   (c) $\int (4x^2 + 9)^{-1/2} \, dx$

3. Evaluate again.

   (a) $\int x^2 \ln x \, dx$
   (b) $\int x^2 e^{2x} \, dx$
   (c) $\int x \tan^2 x \, dx$

4. Evaluate the following integrals, where a, b, c are constants and $a > 0$.

   (a) $\int a^{bx} \, dx$
   (b) $\int \frac{dx}{bx+c}$
   (c) $(bx + c)^n \, dx \qquad (n \neq -1)$
   (d) $\int \frac{\ln(ax)}{bx} \, dx$
   (e) $\int \frac{dx}{a^2+(bx)^2}$
   (f) $\int \tan(bx + c) \, dx$

5. Evaluate each of the following integrals using the substitution $x = \sin\theta$. In the last three problems, check by using another method.

   (a) $\int \frac{dx}{\sqrt{1-x^2}}$
   (b) $\int \sqrt{1 - x^2} \, dx$
   (c) $\int \frac{dx}{(1-x^2)^{3/2}}$
   (d) $\int (1 - x^2) \, dx$
   (e) $\int \frac{x \, dx}{\sqrt{1-x^2}}$
   (f) $\int x(1 - x^2)^{3/2} dx$

PDP Math 1A　　WS#28　　Fall 1987

1. Find these antiderivatives.

   (a) $\int (1 + \sin x)/\cos^2 x \, dx$
   (b) $\int \cos^3(x/2) \sin x \, dx$
   (c) $\int 1/(1 + e^x) \, dx.$
   (d) $\int (x^x + x^x \ln x) \, dx$
   (e) $\int 1/[\sqrt{x}\sqrt{1 - x}] \, dx$
   (f) $\int 1/(1 - x^2) \, dx$
   (g) $\int x \sin x \, dx$
   (h) $\int x^2 \ln x \, dx$
   (i) $\int x^3 e^{2x} \, dx$

2. Evaluate:

   (a) $\int 1/\sqrt{9-4x^2}\ dx$
   (b) $\int x^2 \exp(2x^3)\ dx$
   (c) $\int (x^2+2x+1)/(x^2+1)\ dx$
   (d) $\int 2^{\ln x}/x\ dx$

3. In some cases, integration by parts can be used when there is only one "part". ' Evaluate these two integrals.

   (a) $\int \operatorname{arcsec} x\ dx$
   (b) $\int \ln x\ dx$

4. An integral can sometimes be evaluated by carrying out integration by parts twice:

$$\int e^{ax} \sin bx dx$$

5. Evaluate:

   (a) $\int \sqrt{x^2-1}\ dx$
   (b) $\int \frac{\sqrt{x^2-1}}{x}\ dx$
   (c) $\int \frac{dx}{x\sqrt{x^2-1}}$
   (d) $\int \sqrt{x^2+1}\ dx$
   (e) $\int \frac{dx}{(x^2+1)^{3/2}}$
   (f) $\int \frac{dx}{\sqrt{x^2+1}}$