1. Below is a list of some "simple" algebra problems. Some of the solutions are correct and some of them are **wrong**! For each problem:

   (a) determine if the answer is correct;

   (b) determine if there are any mistakes made in solving the problem and list them (**note** that just because the answer is correct does not mean there are no mistakes);

   (c) if the answer to (A) and/or (B) is **NO**, redo the problem correctly; if the answers to (A) and (B) are **YES**, devise another correct method to solve the problem.

$$a)\ \frac{x^2-1}{x+1} = \frac{x^2+(-1)}{x+1} = \frac{x^2}{x} + \frac{-1}{1} = x-1$$

$$b)\ (x+y)^2 - (x-y)^2 = x^2+y^2-x^2-y^2 = 0$$

$$c)\ \frac{9(x-4)^2}{3x-12} = \frac{3^2(x-4)^2}{3x-12} = \frac{(3x-12)^2}{3x-12} = 3x-12$$

$$d)\ \frac{x^2y^5}{2x^{-3}} = x^2y^5\cdot 2x^3 = 2x^6y^5$$

$$e)\ \frac{(2x^3+7x^2+6)-(2x^3-3x^2-17x+3)}{(x+8)+(x-8)}$$

$$= \frac{(2x)^2-17x+9}{2x} = 2x-17x+9 = -15x+9 = -6x$$

$$f)\ \frac{x^{-1}+y^{-1}}{x^{-1}-y^{-1}} = \frac{(x+y)^{-1}}{(x-y)^{-1}} = (\frac{x+y}{x-y})^{-1} = -\frac{x+y}{x-y} = \frac{x+y}{y-x}$$

2. Solve the following inequalities. Whenever possible, use distance arguments.

a) $\lvert x-3\rvert < 5$	*b)* $0 < \lvert x-2\rvert < 1$
c) $1 < \lvert x+1\rvert \le 2$	*d)* $-1 \le \lvert x+1\rvert \le 2$
e) $0 \le \lvert x+1\rvert \le \infty$	*f)* $0 \le \lvert\lvert x-1\rvert-1\rvert < 1$
g) $\lvert x^2+2\rvert < \lvert x^2+1\rvert$	*h)* $1 < \lvert a-1\rvert + \lvert a-2\rvert < 2$

3. True or False: If $x < 0$, then $|x| = -x$. If false, give an example that illustrates your conclusion. If true, explain why it is true. *Remember that giving an example that works doesn't prove it is true in general.*

4. (a) Show that the expression $\frac{(a+b+|a-b|)}{2}$ is always equal to the larger of the two numbers a and b.

   (b) Write a similar expression for the smaller of two numbers a and b.

5. Solve the inequality and graph the solution on the real line.

$a)\ (x-4)(x+1) < 0$	$b)\ (x-4)(x+1) < -6$	$c)\ x^2 < 1$
$d)\ x(x-3)^4 < 0$	$e)\ x^3 < 1$	$f)\ x^2 < 0$
$g)\ x > \dfrac{1}{x}$	$h)\ x^4 - x \leq 0$	$i)\ x^2 + x + 1 < 0$

6. Use absolute value to define the following intervals.

$$a)\ -7 \leq x \leq 7 \qquad b)\ x < -3 \ \text{ and } \ x > 3 \qquad c)\ -7 < x < 3$$

7. Use absolute value notation to describe:

   (a) The distance between x and 5 is at least 6.

   (b) 6 is at most 3 units from x.

   (c) The distance between x and 7 is greater than 2

   (d) y is closer to 5 than y is to -6.

8. A cake has dimensions $15'' \times 15'' \times 3''$. It is frosted on the sides and top. How can it be divided into 5 pieces so that each piece has the same amount of cake and the same amount of frosting? What if the dimensions are changed? What about 6 pieces? 7 pieces? n pieces?

1. (a) Write an equation which gives the set of all points 2 units from the point $(1, 1)$.
   (b) Write the equation of a circle centered at (a, b), with radius r.
   (c) Write the inequality that represents the set of all points inside a circle centered at $(-1, 2)$ with radius 2.
   (d) Find the equation of a circle whose diameter has endpoints $(8, -5)$ and $(6, 1)$.

2. (a) Find the distance between the points $(-5, -1)$ and $(2, 2)$.
   (b) Find the midpoint of the line segment joining $(-5, -1)$ and $(2, 2)$.
   (c) Identify the type of triangle that has $(-5, -1)$, $(2, 2)$, and $(0, -3)$ as vertices. *(i.e. scalene, right, equilateral, isosceles, etc.)*

3. (a) Find x so that the distance between the points $(6, -1)$ and $(x, 9)$ is 12.
   (b) Find the point $(x, 0)$ that is equidistant from $(6, 1)$ and $(-2, 5)$.
   (c) Find the relationship between x and y so that the point (x, y) is equidistant from the two points $(-2, 1)$ and $(2, -4)$.

4. Find the center of the following circles.

   a) $x^2 + y^2 - 2x + 8y - 20 = 0$ *b)* $4x^2 + 4y^2 - 2x + 8y - 4 = 0$
   c) $x^2 + y^2 - 4x + 6y - 36 = 0$ *d)* $4x^2 + 4y^2 - 24x + 8y + 39$

5. (a) On a Cartesian plane plot the points $(1, 3)$ and $(2, 4)$. On the same plane draw (or plot) the set of points with are equidistant from $(1, 3)$ and $(2, 4)$.
   (b) Find an equation for the set determined in part (a).
   (c) Find the distance from the point $(4, 3)$ to the line $y = 3x + 1$.
   (d) Find an equation for the set of points which are equidistant from the point $(3, 1)$ and the line $y = 2$. **(Don't be sloppy!)**

6. Let L be a line with slope m, $m \neq 0$. Characterize geometrically all lines that have slope:

   $$a)\ m \qquad b)\ -m \qquad c)\ -\frac{1}{m} \qquad d)\ \frac{1}{m}$$

   { *For example, the answer to (a) is : all lines parallel to* L}

7. (a) Show that if a relation is symmetric both with respect to the $y-axis$ and to the origin, then it is also symmetric with respect to the $x-axis$. (You may have to refresh your memory as to what the respective symmetries imply about a relation. Then give a logical, mathematically correct proof!)

   (b) Let f be defined for all x in $\Re$. Let $g(x) = \frac{[f(x)+f(-x)]}{2}$; $h(x) = \frac{[f(x)-f(-x)]}{2}$. Show that g is symmetric with respect to the $y-axis$, h is symmetric with respect to the origin, and that an arbitrary function can always be represented as the sum of two functions, one symmetric with respect to the $y-axis$, the other symmetric with respect to the origin.

8. A cake has dimensions $15'' \times 15'' \times 3''$. It is frosted on the sides and top. How can it be divided into 5 pieces so that each piece has the same amount of cake and the same amount of frosting? What if the dimensions are changed? What about 6 pieces? 7 pieces? n pieces?

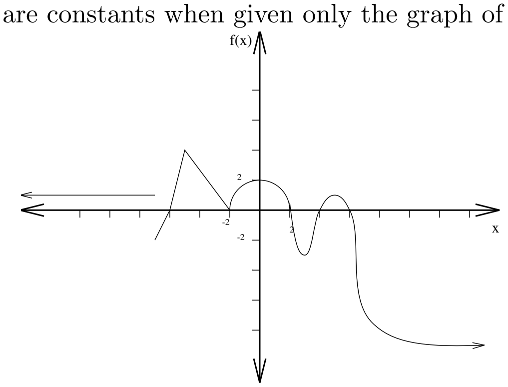
ESP Math 408C   WS#3   Fall 1993

1. The graph of $f(x)$ is given below. Use it to graph the following.

a) $f(3x)$	*b)* $f(-x)$	*c)* $f(-2x)$	*d)* $f(x-1)$
e) $f(x)+1$	*f)* $5f(x)$	*g)* $f(x+2)$	*h)* $5f(3x+2)+1$

2. (a) In your own words, describe the manner in which the graph of $f(x)$ changes when we: multiply $f(x)$ by a constant; add a constant to $f(x)$; multiply x by a constant; add a constant to x.

   (b) Describe the process of drawing $u(x) = af(bx+c)+d$, where $a,\ b,\ c,$ and d are constants when given only the graph of $f(x)$.



3. (a) Determine the domain of $g(z) = [1-(z^2-1)^{\frac{1}{2}}]^{\frac{1}{2}}$

   (b) Find the domain, range, asymptotes, and symmetries of the relation $\frac{y+7}{y-7} = \frac{x+3}{x-3}$.

   (c) What are the asymptotes of $y = \frac{x}{|x|+2x+1}$?

   (d) Prove that the graph of $1 + x^2 + x^3 = y^2$ is not symmetric with respect to the origin.

   (e) Is the graph of $\{\ (x,y) \mid (2+y^3)(x-1) = (2-y^3)(x+1)\}$ symmetric with respect to the origin?

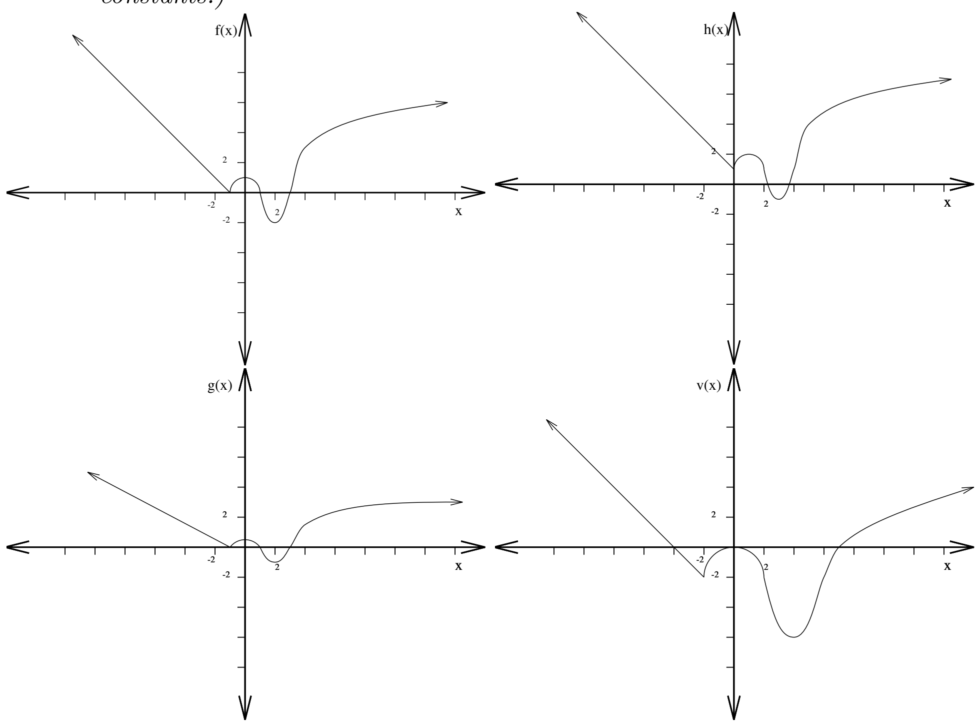
4. Let S be the relation $\{\ (x,y) \mid (x^2-4)y = x^2+2x\}$.

   (a) Using set notation, describe the domain of S and the range of S.

   (b) Find all horizontal and vertical asymptotes of S and describe each by an equation.

   (c) Find the numbers a and b such that the point (a,b) is on the graph of S but the point $(-a,b)$ is not. Then complete the following sentence: "The fact that (a,b) is on the graph but $(-a,b)$ is not shows that the graph of S is not symmetric about _______."

5. Given the graph of $f(x)$, provide an expression for the following functions $h(x)$, $g(x)$, and $v(x)$ in terms of $f(x)$. *(A function $u(x)$, say, written in terms of $f(x)$ would be $u(x) = af(bx + c) + d$, where a, b, c, and d are constants.)*



6. Given that $f(x)$ is a function that has x-intercepts at $x = 0$, $x = 2$, and $x = -2$; y-intercept $(0, 0)$; horizontal asymptote $y = -1$; and vertical asymptotes at $x = 3$ and $x = -3$.

   (a) Graph $f(x)$. Is your graph unique? If so, why? If not, how many distinctive shapes can your graph have?

   (b) As $x \to 3^+$ then $f(x) \to$ ____?

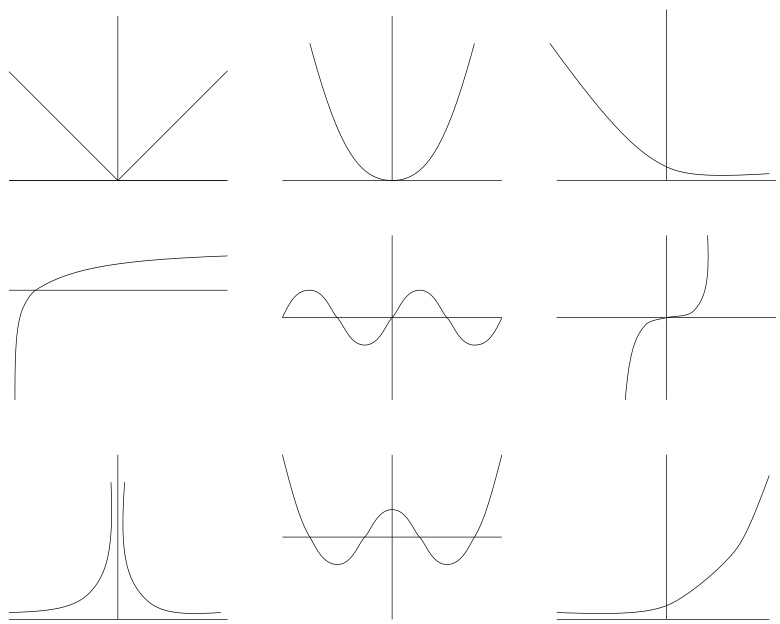
   (c) As $x \to 3^-$ then $f(x) \to$ ____?

   (d) As $x \to -3^+$ then $f(x) \to$ ____?

   (e) As $x \to -3^-$ then $f(x) \to$ ____?

7. Which of the graphs below could correspond to the following functions.
Match them.

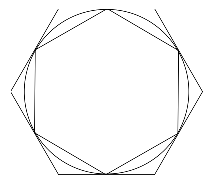
a) odd power function	b) even power function	c) a quadratic
d) $\log x$	e) a 4th degree polynomial	f) b^x
g) $\sin x$	h) $(\frac{1}{2})^x$	i) absolute value



1. Fill in the tables, (use calculators if necessary) then guess what the *last* two boxes should be. (Sorry: no tables yet.)

2. For each of the tables in (1) write it as a limit; i.e. $\lim_{x\to a} f(x) = L$ where you fill in $f(x)$, a, and L.

3. What is the ratio of the areas of the inner and outer hexagons in this figure?



4. (a) For $n = 3, 4, 5$ find the perimeter of an n-sided regular polygon (i.e. an equilateral triangle, a square, and a regular pentagon) inscribed in a circle of radius 2.

   (b) Find a general formula for an n-sided regular polygon's perimeter (inscribed in a circle of radius 2).

   (c) Draw a picture of a circle of radius 2 with a twelve and a 20-sided regular polygon inscribed.

   (d) What is the limit, as $n \to \infty$ of the formula you found in part (b).

5. Let
$$f(x) = \frac{x^2 - 1}{x + 1}$$

   (a) Sketch the graph of $f(x)$, and determine its domain and range.

   (b) Evaluate $f(-1)$ and $f(1)$.

   (c) Evaluate
$$\lim_{x\to -1} f(x) \qquad \text{and} \qquad \lim_{x\to 1} f(x).$$

6. Given that $f(x)$ is a function that has x-intercepts at $x = 0$, $x = 2$, and $x = -2$; y-intercept $(0, 0)$; horizontal asymptote $y = -1$; and vertical asymptotes at $x = 3$ and $x = -3$.

   (a) Graph $f(x)$. Is your graph unique? If so, why? If not, how many distinctive shapes can your graph have?

   (b) As $x \to 3^+$ then $f(x) \to$ ____?

   (c) As $x \to 3^-$ then $f(x) \to$ ____?

   (d) As $x \to -3^+$ then $f(x) \to$ ____?

   (e) As $x \to -3^-$ then $f(x) \to$ ____?

7. Compute the following limits (think about the *theorems* you use).

$$a)\ \lim_{x\to 1}\frac{x^3-1}{(x-1)^2} \qquad b)\ \lim_{x\to -2}\frac{x^3+8}{x+2} \qquad c)\ \lim_{x\to 1}\frac{x^3-2x^2+2x-1}{x^3-1}$$

$$d)\ \lim_{x\to 8}\frac{\sqrt[3]{x}-2}{x-8} \qquad e)\ \lim_{x\to -1}\frac{\frac{1}{x}+1}{x+1} \qquad f)\ \lim_{x\to a}\frac{x^n-a^n}{x-a}$$

$$g)\ \lim_{x\to 2}\frac{\sqrt{2}-\sqrt{x}}{2-x} \qquad h)\ \lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x^2} \qquad i)\ \lim_{x\to 4}\frac{\sqrt[3]{x}-\sqrt[3]{4}}{x-4}$$

8. Newcomb's Paradox: There are two boxes before you, A and B. You may open both boxes, or else just open B. You may keep what is inside any box you open, but you may not keep what is inside a box you do not open. A very powerful being, who has been invariably accurate in his predictions about your behavior in the past has already acted in the following way:

– He has put \$1,000 in box A.

– If he has predicted that you will open just box B, he has in addition put \$1,000,000 in box B.

– If he has predicted that you will open both boxes, he has put nothing in B.

(a) Give an argument to show that the best strategy is to open both boxes.

(b) Give an argument to show that the best strategy is to open only box B.

(c) Where is the crux of this paradox?

ESP Math 408C   WS#5   Fall 1993

1. Sketch the functions $g(x),\ h(x),$ and $f(x)$ such that the following conditions hold.

   a) $\lim_{x\to 5} g(x) = 10$ however $g(5)$ is not defined.

   b) $\lim_{x\to 2} h(x) = 3$ however $h(2) \neq 3$.

   c) $\lim_{x\to 0} f(x)$ does not exist, however $f(0) = 2$.

2. Sketch a graph of a function which satisfies all of the given properties. (Remember, not all functions are continuous.)

   a) $\lim_{x\to 0} f(x) = 0,\ \ f(0) = 10,\ \ \lim_{x\to 1^+} f(x) = -1,\ \ \lim_{x\to 1^-} = 1,\ \ f(1) = 0.$

   b) $\lim_{x\to 2^-} g(x) = \infty,\ \ \lim_{x\to 2^+} g(x) = -\infty.$

   c) $\lim_{x\to n^-} h(x) = n,\ \ h(n) = n+1,$ for every integer n.

3. (a) Complete Pascal's Triangle to the seventh level.

   0th column
   1st column
   0th level → 1
   1st level → 1   1

   (b) Notice something interesting about it!

   (c) Ex. Expand $(x+a)^7$ without multiplying it out.

   (d) Ex. What is $1+2+3+4+5+6$? *(Hint: Use the "L" theorem.)*

   (e) Ex. How many different combinations of 3 people can you choose from a group of 5? *(Hint: 5th level, 3rd column)*

4. Complete the following table.

n	expand $(x+a)^n$	factor $x^n - a^n$	factor (if possible): $x^n + a^n$
2			
3			
4			

5. For each below do the following: Compute the limits, write in words what techniques you used to solve them, then write another limit of the same type.

$$a)\ \lim_{x\to 1}\frac{x^3-x^2+x-1}{x^2-1} \qquad \lim_{x\to 0}\frac{(1+x)^2-1}{(1+x)^3-1}$$

$$b)\ \lim_{x\to 4}\frac{\sqrt[3]{x}-\sqrt[3]{4}}{x-4} \qquad \lim_{x\to 0}\frac{(x-2)+2}{\sqrt[3]{x-2}+\sqrt[3]{2}}$$

$$c)\ \lim_{x\to 3}\frac{2x^3-18x^2+54x-54}{x-3} \qquad \lim_{x\to -1}\frac{x^4+4x^3+6x^2+4x+1}{(x^2-1)^4}$$

$$d)\ \lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3} \qquad \lim_{x\to 0}\frac{x^2}{1-\sqrt{1-x^2}}$$

$$e)\ \lim_{x\to a}\frac{|x-a|}{x-a} \qquad \lim_{x\to -3^-}\frac{|x+3|}{x+3}$$

$$f)\ \lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1} \qquad \lim_{x\to 1}\frac{\sqrt[4]{x}-1}{x-1}$$

$$g)\ \lim_{x\to 2} f(x),\ \lim_{x\to 2^-} f(x),\ \lim_{x\to 2^+} f(x) \qquad where\ f(x)=\begin{cases} x^2, & x\ge 2\\ \frac{x^2-4}{x-2}, & x<2\end{cases}$$

6. Sketch the graphs of the following functions, determine the points of discontinuity, and state whether it's a removable or nonremovable (i.e. essential) discontinuity.

$$a)\ f(x)=\frac{x^2-1}{x+1} \qquad b)\ f(x)=\begin{cases} x^2 & ,\ x<-2\\ -2 & ,\ x=-2\\ x & ,\ -2<x\le 0\\ 2 & ,\ 0<x<2\\ 3x+1 & ,\ 2\le x\end{cases} \qquad c)\ f(x)=|x+2|$$

7. (a) State the Squeeze Theorem. *(Sometimes known as the Pinching Theorem.)*

   (b) Use part (a) to find $\lim_{x\to c} f(x)$ given that

$$\left|\frac{f(x)-f(c)}{x-c}\right| \le M \text{ for } x\ne c.$$

8. <u>The light coin</u>
   You are given 8 coins and told that one is not up to standard weight (but all the rest are). You have a pair of balance scales. How can you proceed to identify the light coin in only two weighings?

1. Recall that $\sin^2 x = 1 - \cos^2 x$. Write an expression for each trig function squared in terms of one of the other trig functions.

   a) $\cos^2 x$ *b)* $\tan^2 x$ *c)* $\cot^2 x$ *d)* $\sec^2 x$ *e)* $\csc^2 x$

2. Compute the limits.

   a) $\lim_{x\to 0} \dfrac{\sin^2 2x}{x^2}$ *b)* $\lim_{x\to 0} \dfrac{1-\cos 3x}{x^2}$ *c)* $\lim_{x\to 0} \dfrac{x}{1-\sec x}$

   d) $\lim_{x\to 0} \dfrac{1+\cot^2 3x}{4x^2}$ *e)* $\lim_{x\to \frac{\pi}{6}} [1-\sec(x+\dfrac{\pi}{3})](x-\dfrac{\pi}{6})^2$ *f)* $\lim_{x\to \frac{\pi}{2}} \dfrac{\cos x}{2x-\pi}$

3. Find an appropriate $f(x)$ for each of the following.

   a) $\lim_{x\to 2} f(x) = 0,$ $f(2) = \dfrac{0}{0}$ *b)* $\lim_{x\to -1} f(x) = 2,$ $f(-1) = \dfrac{0}{0}$

   c) $\lim_{x\to \pi} f(x) = \infty,$ $f(\pi) = \infty \cdot 0$ *d)* $\lim_{x\to 1} f(x) = 1,$ $f(1) = \dfrac{0}{0}$

   e) $\lim_{x\to \pi} f(x) = 0,$ $f(\pi) = \infty \cdot 0$ *f)* $\lim_{x\to d} f(x) = 3,$ $f(d) = \infty \cdot 0$

   Do you notice anything strange about the limit of $f(x)$ as $x \to c$ when $f(c)$ has the form $\infty \cdot 0$?

4. Let $f(x) = \frac{1}{x}\sin x$, assume we only take $x > 0$.

   (a) Sketch the graph of $f(x)$.

   (b) Find $\lim_{x\to 0} f(x)$ and $\lim_{t\to 0} f(\frac{1}{t})$.

5. Let $f(x) = 3x^4 - 4x^3$.

   (a) Where is $f(x) = 0$?

   (b) What does the Intermediate Value Theorem say about $f(x)$ in $(-\infty, 0)$? In $(0, \frac{4}{3})$? In $(\frac{4}{3}, \infty)$?

   (c) Where is $f(x) > 0$? Where is $f(x) < 0$?

   (d) Determine a procedure for using the Intermediate Value Theorem to decide the intervals in which a function is positive and those in which it is negative.

6. (a) Show that the equation $\sin x - \frac{1}{2}x = 0$ has a root in $[1, 2]$.
   (b) Is the root in $[1, 1.5]$ or in $[1.5, 2.0]$?
   (c) Continue in this manner until you have located the root in an interval of length 18. *(Hint: Recall the Intermediate Value Theorem.)*

7. Show that if $0 \leq f(x) \leq 1$ for each $x \in [0, 1]$ and $f(x)$ is a continuous function, then there is some number a, $0 \leq a \leq 1$, such that $f(a) = a$.

8. More limits!

a) $\displaystyle\lim_{x\to 0} \frac{\tan 3x}{x}$	*b)* $\displaystyle\lim_{x\to 0} \frac{\sec 2x \tan 2x}{x}$	*c)* $\displaystyle\lim_{t\to 0^+} \sin(\sqrt{x}x)$
d) $\displaystyle\lim_{x\to 0} \frac{1-\cos 2x}{x}$	*e)* $\displaystyle\lim_{x\to 0} \frac{1-\cos 2x}{x^2}$	

9. A revolving beacon from a lighthouse shines on the straight shore, and the closest point on the shore is a pier one half mile from the lighthouse. Let θ denote the positive acute angle between the shore and the beam of light. Write the distance from the pier to the point where the light shines on the shore as a function of θ.

10. For each of the following, define a function satisfying the conditions, graph your function.
    (a) $g(x)$ is a rational function which is defined for all x **except** $x = 1, 2$.
    (b) $\lim_{x\to 0} f(x)$ does not exist and $|f(x)| < 2$ for all x.
    (c) $f(x)$ is discontinuous at every point.

1. True or False. If false, give a counter-example.

   (a) If $f(-1) = -1$ and $f(1) = 1$, then $f(0) = 0$.

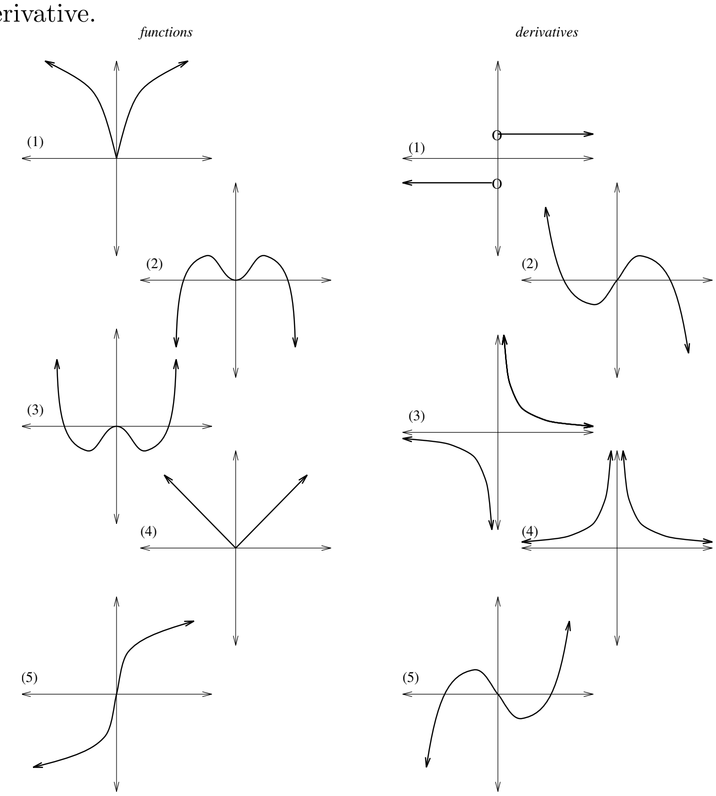
   (b) If $f(-1) = -1$ and $f(1) = 1$, then there is a point c, such that $-1 < c < 1$ and $f(c) = 0$.

   (c) If $f(-1) = -1$, $f(1) = 1$, and $f(x)$ is continuous, then there is a point c, $-1 < c < 1$, such that $f(c) = 0$.

   (d) If $f(0) = 0$, $f(1) = 10$ and $f(x)$ is continuous, then on the interval $[0, 10]$, $f(x)$ must have a maximum or a minimum value.

   (e) If $f(x)$ is continuous at a point a then it is differentiable there.

2. Match each of the derivatives shown below with the corresponding functions. Which functions shown below are continuous? Which derivatives shown below are continuous? Explain what shapes continuous functions must have to produce jump discontinuities in the derivative and asymptotes in the derivative. Draw a continuous function whose derivative has a jump discontinuity at $x = 1$ and an asymptote at $x = 5$; draw the derivative.



3. Suppose $f(x)$ is a function that is differentiable for all real x, and $f(a+b) = f(a)f(b)$ for all real a and b with $f'(0) = 1$ and $f(0) = 1$.
   (a) Show that $f(x) = f'(x)$ for all real x.
   (b) Draw a picture of such a function $f(x)$.
   (c) Suppose we don't know that $f'(0) = 1$. What can you say about $f(x)$ and $f'(x)$?

4. Imagine a road on which the speed limit is specified at every single point. In other words, there is a certain function L such that the speed limit x miles from the beginning of the road is $L(x)$. Two cars A and B, are driving along this road; car A's position at time t is $a(t)$, and car B's is $b(t)$.
   (a) What equation expresses the fact that the car A always travels at the speed limit? (Hint: The the answer is *not* $a'(t) = L(t)$.)
   (b) Suppose that A always goes at the speed limit, and that B's position at time t is A's position at time $t-1$. Show that B is also going at the speed limit at all times.
   (c) Suppose B always stays at constant distance behind A. Under what conditions will B still always travel at the speed limit?

5. Prove, starting from the definition (and drawing a picture to illustrate):
   (a) if $g(x) = f(x) + c$, then $g'(x) = f'(x)$;
   (b) if $g(x) = cf(x)$, then $g'(x) = cf'(x)$.

6. The function $s(t) = \alpha t^2 + \beta t + \gamma$ gives the position of a free-falling object at time t.
   (a) Find the average rate of change in distance with respect to time (**average velocity**) over the interval $[t_1, t_2]$.
   (b) The instantaneous rate of change in distance with respect to time (**velocity**) is given by $v(t) = s'(t) = 2\alpha t + \beta$. Find the instantaneous rate of change in velocity with respect to time. Do you know another name for this quantity?
   (c) The position function given is quadratic in t. Can you make any generalizations about what a quadratic position function implies about the velocity and acceleration of an object? For free-falling objects, we can be sure that $|\alpha| = \frac{\text{acceleration}}{2} = 16\frac{\text{ft}}{\text{sec}^2}$. Why?

7. A giddily gleeful student, elated over passing a Calculus 408C examination, hurls a somewhat large calculus book directly upward from the ground. It moves according to the law $s(t) = 96t - 16t^2$ where t is the time in seconds after it is thrown and $s(t)$ is the height in feet above the ground at time t. Find:

   (a) the velocity of the book after 1.5 seconds;

   (b) the maximum height the book reaches;

   (c) the average speed of the book during its upward rise;

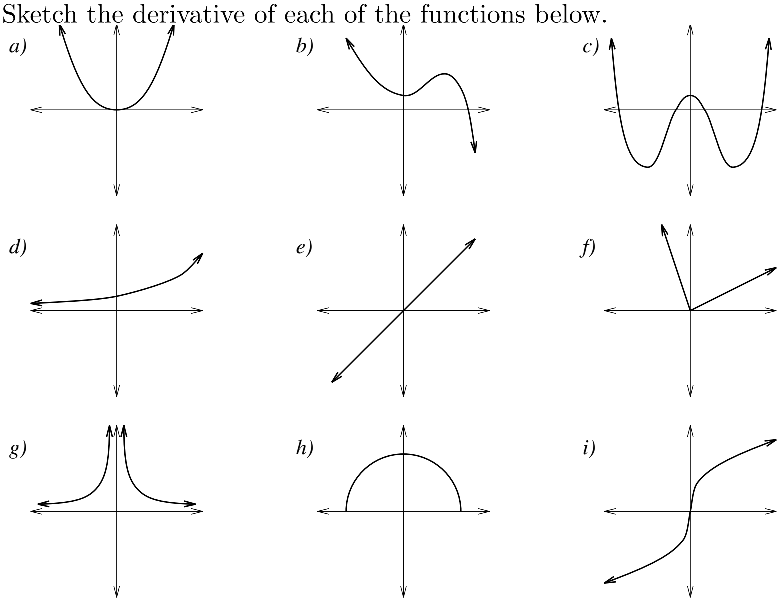
   (d) the acceleration of the book at its maximum height;

   (e) the rate of change of the acceleration of the book after 4 seconds;

   (f) the time it would take for the 6 ft. tall student to have the misfortune of being hit on the head by the book.

ESP Math 408C WS#8 Fall 1993

1. Sketch the derivative of each of the functions below.



2. (a) Find a continuous function $f(x)$ such that $f(-1) = f(2) = f(5) = 0$ and $f(0) \neq 0$.

   (b) Find a function that is not continuous at $x = 0$, but is continuous for all $x \neq 0$.

   (c) Find a function that is continuous at $x = 0$ but is not continuous at any other x.

   (d) Find a function that is continuous at each irrational number and not continuous at any rational numbers.

3. Which value(s) of a make the function $f(x)$ continuous? differentiable?

$$f(x) = \begin{cases} ax^2 & \text{for } \ x \geq 2 \\ \frac{a^2}{4}x^2 + 3 & \text{for } \ x < 2 \end{cases}$$

4. Prove that $f(x) = x^3 - 3x + c$ never has two roots in $[0, 1]$ no matter what c is.

5. Prove that it is impossible to find two differentiable functions $f(x)$ and $g(x)$ for which $f(0) = g(0) = 0$ and which satisfy $f(x)g(x) = x$ for all x. *(Hint: differentiate)*

6. (a) Prove that Galileo was wrong: if a body falls a distance $s(t)$ in t seconds, and $s'(t)$ is proportional to s, then s cannot be a function of the form $s(t) = ct^2$.

   (b) Prove that the following facts are true about s if $s(t) = (\frac{a}{2})t^2$ *(the first fact will show why we switched from c to $\frac{a}{2}$)*:
   (i) $s''(t) = a$ (the acceleration is constant).
   (ii) $[s'(t)]^2 = 2as(t)$.

   (c) If s is measured in feet, the value a is 32. How many seconds do you have to get out of the way of a chandelier which falls from a 400-foot ceiling? If you don't make it, how fast will the chandelier be going when it hits you? Where was the chandelier when it was moving with half that speed?

7. (a) Suppose that $f(x)$ is differentiable at x. Prove that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

   Hint: Remember an old algebraic trick...a number is not changed if the same quantity is added to and then subtracted from it.

   (b) Prove, more generally that

$$f'(x) = \lim_{h,k \to 0^+} \frac{f(x+h) - f(x-k)}{h+k}.$$

8. If $f + g$ is differentiable at a, are f and g necessarily differentiable at a? If $f \cdot g$ and f are differentiable at a, what conditions on f imply that g is differentiable at a?

1. Let $f(x) = x^3 + 3x^2 - 3x$.
   (a) At any point (x_0, y_0) on the graph, what is the slope of the tangent line to the graph?
   (b) The graph of $f(x)$ has two tangent lines parallel to the line $y = 6x + 100$. Find the equations of these two lines.

2. (a) Find all points on the graph of $y = x^2$ whose tangent lines pass through the point $(5, 0)$.
   (b) Show that no line tangent to the graph of $f(x) = x + \frac{1}{x}$ passes through the origin.

3. Decide where each of the following functions is continuous.

$a)\ f(x) = \sqrt[3]{x}$ $\quad b)\ f(x) = x^2$ $\quad c)\ f(x) = |4 - x^2|$

$$d)\ f(x) = \frac{x^2}{x^2+1} \qquad e)\ f(x) = \begin{cases} \frac{1-\frac{1}{x}}{1-\frac{1}{x^2}} & \text{if } x \neq 1 \\ \frac{1}{2} & \text{if} x = 1 \end{cases} \qquad f) f(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

$$g)\ f(x) = \frac{x+2}{x^2-4} \qquad h)\ f(x) = \begin{cases} 0 & \text{if } x \in \mathbf{Q} \\ x & \text{if } x \in \mathbf{R\text{-}Q} \end{cases} \qquad i)\ f(x) = \frac{1}{x+3} + \frac{1}{|x+3|}$$

4. (a) Find the derivatives of the functions in 1 a),b),c),d),g), and i).
   (b) Graph the derivatives found in part (a) and their corresponding functions.

5. (a) Compute $g'(1)$, where $g(x) = \frac{x+f(x)}{x-f(x)}$, $f(1) = 4$, and $f'(1) = 2$.
   (b) Suppose $f(x) = xf'(x)$ for all x in the domain of $f(x)$ then what can we say about the function $g(x)$?

6. Show that $f(x) = x^3 - 3x^2 + 3x - 6$ has exactly one real root.

7. Prove that it is impossible to find two differentiable functions $f(x)$ and $g(x)$ for which $f(0) = g(0) = 0$ and which satisfy $f(x)g(x) = x$ for all x. *(Hint: differentiate)*

8. Derive the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$ when given only that
$$\frac{\mathrm{d}\sin x}{\mathrm{dx}} = \cos x \text{ and } \frac{\mathrm{d}\cos x}{\mathrm{dx}} = -\sin x.$$

9. (a) Show that
$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$
   (b) Use the above fact to calculate:

$a)$ $\lim_{x\to 8} x^{\frac{4}{3}} - 16x - 8$ $\quad$ $b)$ $\lim_{x\to 2}(6+x)^{\frac{2}{3}} - 4x - 2$ $\quad$ $c)$ $\lim_{x\to 1}\frac{x^{100}-1}{x-1}$

$d)$ $\lim_{x\to -1}\frac{x^2+x}{(x+2)(x+1)}$ $\quad$ $e)$ $\lim_{x\to 1}\frac{x^3-2x^2+2x-1}{x-1}$

10. In the decimal addition $SAGE + SUAVE + SAGE = 46933$ the addend digits have been replaced unambiguously by letters. Restore the digits and identify who is described.

1. Write $f(x)$ as a composition of two functions in two different ways. Write $f(x)$ as a composition of three functions. Differentiate $f(x)$.

   a) $f(x) = \sqrt{x^2+1}$ *b)* $f(x) = \dfrac{1}{\sqrt{3x+2}}$ *c)* $f(x) = (2x^2 - x)^{\frac{5}{2}}$

2. Assume that $f(x)$ is a differentiable function and that the values of $f(x)$ and its derivative at the points $x = 0, 1, 2$, and 3 are given by:

$$\begin{array}{llll} f(0) = 3 & f(1) = 5 & f(2) = -2 & f(3) = 6 \\ f'(0) = -1 & f'(1) = 0 & f'(2) = 3 & f'(3) = 1 \end{array}$$

   Let $g(x) = x^2 - 3x + 2$. For each function below calculate the derivative at the given point.

   a) $\dfrac{f(x)}{g(x)}$ $x = 0$ *b)* $f(x)g(x)$; $x = 1$

   c) $f(g(x))$; $x = 2$ *d)* $g(f(x))$; $x = 3$

3. Differentiate the following.

   a) $f(x) = (1 + \sqrt{x})^{\frac{1}{2}}$ *b)* $g(x) = [(x^2+1)^2 + (x^2+1) + 1]^2$

   c) $f(x) = [x - \dfrac{2}{x + \sin x}]^{-1}$ *d)* $f(x) = \sin(\dfrac{\cos x}{x})$

   e) $f(x) = (\sin^2 x)(\sin x^2)(\sin^2 x^2)$ *f)* $f(x) = \dfrac{\sin(\cos x)}{x}$

4. (a) Prove that the formula for the derivative of an inverse function is

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

   (Hint: Let $g(x) = f^{-1}(x)$, then $f(g(x)) = x$. Differentiate.)

   (b) Find $f^{-1}(x)$ given that $f(x) = \frac{2x-3}{x+2}$.

   (c) Differentiate $f^{-1}(x)$ from part b) and compare with the derivative you get by applying the formula you get in part a).

5. Fill in the table, given that $h(x) = f(g(x))$.

a	$g(a)$	$g'(a)$	$f(a)$	$f'(a)$	$h(a)$	$h'(a)$
0	1	-3	-3	-2	$-\frac{1}{3}$	
1	2	0		-7		
2	2	-1	-1			3

6. Sketch the graph for $g(x),\ f(x),$ and $h(x)$ satisfying the table in 5).

7. (a) Suppose that $f(a) = g(a) = h(a)$, that $f(x) \leq g(x) \leq h(x)$ for all x, and that $f'(a) = h'(a)$. Prove that $g(x)$ is differentiable at a, and that $f'(a) = g'(a) = h'(a)$. *(Hint: Begin with the definition of $g'(a)$. The Squeeze (or Pinching) Theorem may be useful.)*

   (b) Show that the conclusion does not follow if we omit the hypothesis $f(a) = g(a) = h(a)$.

   (c) Draw a graph illustrating part a).

   (d) Draw a graph illustrating part b).

8. (a) The radius of a balloon is given by the formula $r(t) = \sqrt{t^2+1} - 1$. *(Imagine you are blowing it up, so after one second, the radius is $r(1) = \sqrt{2} - 1$ inches.)* Give a formula for the rate of change in the radius with respect to time.

   (b) Give a formula for the rate of change of the volume of the balloon with respect to time.

   (c) How fast is the volume changing when $t = 2$?

   (d) How fast is the volume changing when $r = 2$?

9. Find $f'(x)$ in terms of $g(x)$ and $g'(x)$, where $g(x) > 0$ for all x. *(Recall: If c is a constant, then $g(c)$ is a constant.)*

a) $f(x) = g(x)(x-a)$	*b)* $f(x) = g(a)(x-a)$	*c)* $f(x) = g(x+g(x))$
d) $f(x) = \dfrac{g(x)}{x-a}$	*e)* $f(x) = \dfrac{1}{g(x)}$	*f)* $f(x) = g(xg(a))$
g) $f(x) = \sqrt{g(x)^2}$	*h)* $f(x) = \sqrt{g(x^2)}$	*i)* $f(2x+3) = g(x^2)$
		Hint: $x = 2\dfrac{(x-3)}{2} + 3$

10. Find $f'(0)$ if

$$f(x) = \begin{cases} g(x)\sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0, \end{cases}$$

and

$$g(0) = g'(0) = 0.$$

1. Find the equation of the line tangent to the curve $x + y + x^4y^3 = 3$ at the point $(1, 1)$.

2. (a) Find $\frac{\mathrm{dy}}{\mathrm{dx}}$ at the point (x, y) on the curve $x^2 - xy + y^2 = 3$.
   (b) For $x = 0$ and $x = \pm 3$ find the associated values of y and $\frac{\mathrm{dy}}{\mathrm{dx}}$ on the curve $x^2 - xy + y^2 = 9$.
   (c) Find all points on the curve $x^2 - xy + y^2 = 9$ where the tangent is horizontal; then find all the points where it is vertical.
   (d) Sketch the curve $x^2 - xy + y^2 = 9$.
   (e) Find all points of the curve $x^2 - xy + y^2 = 9$ that are closest to and farthest from the origin.

3. For each of the following relations, find $\frac{\mathrm{dy}}{\mathrm{dx}}$ both explicitly and implicitly and verify that these results are in agreement. Use the results to help sketch the graphs.

   $$a)\ x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1 \qquad b)\ x^2 + y^2 = 1 \qquad c)\ xy = 4$$

4. A revolving beacon from a light house shines on the straight shore, and the closest point on the shore is a pier one half mile from the lighthouse. Let θ denote the angle between the lighthouse, pier, and point on the shore where the light shines.

   (a) Write the distance from the pier to the point of light as a function of θ .
   (b) What is the rate of change of the distance from the pier to the point of light with respect to θ.
   (c) Suppose θ is a function of time t. Give an expression for the rate of change of distance with respect to time t.
   (d) Suppose that the light makes 1 revolution per minute. How fast is the light traveling along the straight beach at the instant it pases over a shorepoint 1 mile away from the shorepoint nearest the searchlight?

5. Particle A moves along the positive horizontal axis, and particle B along the graph of
   $$f(x) = -\sqrt{3}x,\ x \leq 0.$$
   At a certain time, A is at the point $(5, 0)$ and moving with speed 3 units/sec; and B is at a distance of 3 units from the origin moving with speed 4 units/sec. At what rate is the distance between A and B changing?

6. A man 6 feet tall walks at the rate of 5 feet per second toward a street light that is 16 feet above the ground. At what rate is the tip of the shadow moving? At what rate is the length of his shadow changing when he is 10 feet from the base of the light?

Review

1. Consider the relation $y^8x^2 = 5$. Find the points where the tangent line to its graph is parallel to the line $2y + 18x = 3$. Find the points where the tangent line to its graph is perpendicular to the line $y + 4x = 2$.

2. Let $f(x) = x^2 + c$.

   (a) For what values of c does the tangent line to the graph of $f(x)$ at $x = 1$ pass through $(3, 5)$?

   (b) For what values of c does the normal line to the graph of $f(x)$ at $x = 2$ pass through $(8, 1)$?

3. For each of the following relations, find $\frac{\mathrm{dy}}{\mathrm{dx}}$ both explicitly and implicitly and verify that these results are in agreement. Use the results to help sketch the graphs.

   a) $x^3 + y^3 = 1$ *b)* $|x| + |y| = 1$

4. A stone is dropped into the water from a bridge 144 ft above the water. Another stone is thrown vertically down 1.0 seconds after the first is dropped. Both stones strike the water at the same time.

   (a) What was the initial speed of the second stone?

   (b) Plot speed versus time on a graph for each stone, taking time, $t = 0$, as the instant the first stone was released.

5. Suppose that $h(x) = g()u(x)$, where $u(x) = x^3 + 1$, and that $g'(1) = 2$, $g'(2) = 4$, $g'(9) = 16$, $g'(13) = 8$. Find $h'(2)$ and $h''(2)$.

6. Differentiate:

   a) $\tan(x^2 + \frac{1}{x^2})$ *b)* $\sqrt{x^3 + 2} + \sin\sqrt{x + \sqrt{x}}$

7. Given that a spherical raindrop evaporates at a rate proportional to its surface area, how fast does the radius shrink?

8. A light at the top of a pole which is h meters high. A ball is dropped from half the height of h at a point which is at a horizontal distance d in meters from the pole. Assume that the ball falls according to the law $s = gt^2$, where t is the time in seconds, s is the distance in meters, and g is a constant. Find how fast the tip of the shadow of the ball is moving along the ground t_0 seconds after it is dropped.

9. Mr. A is walking at 2 mph due south toward point c. Ms. B is walking at 3 mph due east towards point c. At a certain instant Mr. A is 3 miles from c and Ms. B is 4 miles from c.

   (a) At that instant, what is the rate that the distance between Mr. A and Ms. B is decreasing?

   (b) At that instant, what is the rate that the angle $\angle ABC$ is changing?

1. A tank contains 1000 cubic feet of natural gas at a pressure of 5 pounds per square inch. Find the rate of change of the volume if the pressure decreases at a rate of 0.05 pounds per square inch per hour. (Assume Boyle's law: $pressure \times volume = constant$.)

2. A spherical balloon is inflated with gas at the rate of 100 cubic feet per minute. Assuming that the gas pressure remains constant, how fast is the radius of the balloon increasing at the instant when the radius is $a)3ft.$ $b)10ft.$ $c)100ft.$

3. A boat is pulled in to a dock by means of a rope with one end attached to the bow of the boat, the other end passing through a ring attached to the dock at a point 4 feet higher than the bow of the boat. If the rope is pulled in at the rate of 2 feet per second, how fast is the boat approaching the dock when a) 10 feet of the rope are out? b) 5 feet of the rope are out?

4. A balloon is 200 feet off the ground and rising vertically at the constant rate of 15 feet per second. An automobile passes beneath it traveling along a straight road at the constant rate of 45 miles per hour. How fast is the distance between them changing one second later?

5. Two airplanes are flying north at the same height on parallel paths ten miles apart with speeds of 400 and 600 miles per hour. How fast is the distance between the planes changing when the slow plane is five miles further north than the fast one?

6. (a) What does it mean when you say a function is one-to-one on its domain?
   (b) True or False: Every function which is one-to-one has an inverse function.
   (c) True or False: Every function whose derivative is never zero is a one-to-one function.
   (d) True or False: If $f(x)$ is a one-to-one function, then $f'(x)$ is never zero.
   (e) True or False: If $f(x)$ is a one-to-one function, then $f(x)$ has no relative maxima or minima.
   (f) Show that $f(x) = \frac{ax+b}{cx+d}$ has no local extremes if $ad - bc \neq 0$.

7. Two cars, car A traveling east at 30 mph and car B traveling north at 22.5 mph, are heading toward an intersection I. At what rate is the angle IAB changing at the instant when cars A and B are 300 feet and 400 feet, respectively, from the intersection?

8. (a) Rain is falling at the rate of q inches per hour into an open conical tank of height H and radius R. Show that at each instant the rate at which water is rising in the tank is

$$q \times \frac{(\textit{area of tank opening})}{(\textit{area of water surface})}$$

(b) Show that the result of (a) is true for an open tank of arbitrary shape.

1. True or False. If false give a counter-example.

   (a) If $f(x)$ is continuous at $x = 2$ and $f(2)$ is the maximum y-value of the function then $f(x)$ is differentiable at $x = 2$.

   (b) You take a boat trip from New York to London and follow a smooth, but curvy course. At some point on your journey you are traveling parallel to the direct straight line course.

   (c) On a trip from Dallas to Austin you go through Waco at 10 p.m. and Temple at 11 p.m. (50 miles apart). Between 10 and 11, at some point, you were driving exactly 50 *mph*.

   (d) If $f(x)$ is defined on $[0, 1]$ and continuous and differentiable on $(0, 1)$ then there exists a point x_0 in $[0, 1]$ such that $f'(x_0) = f(1) - f(0)$.

2. The following theorem was proved by the French mathematician Rolle, in connection with the problem of approximating roots of polynomials, but the result was not originally stated in terms of derivatives. In fact, Rolle was one of the mathematicians who never accepted the new notions of calculus. This was not such a pigheaded attitude, in view of the fact that for one hundred years no one could define limits in terms that did not verge on the mystic, but on the whole history has been particularly kind to Rolle.

3. Suppose that a and b are two consecutive roots of a polynomial function $f(x)$, but that a and b are not double roots, so that we can write $f(x) = (x-a)(x-b)g(x)$ where $g(a) \neq 0$ and $g(b) \neq 0$.

   (a) Prove that $g(a)$ and $g(b)$ have the same sign. *(Remember that a and b are consecutive roots.)*

   (b) Prove that there is some number x with $a < x < b$ and $f'(x) = 0$. *(Also draw a picture to illustrate this fact.) Hint: Compare the sign of $f'(a)$ and $f'(b)$.*

   (c) Now prove the same fact, even if a and b are multiple roots. *Hint: If $f(x) = (x-a)^m(x-b)^n g(x)$ where $g(a) \neq 0$ and $g(b) \neq 0$, consider the polynomial function*

   $$h(x) = \frac{f'(x)}{(x-a)^{m-1}(x-b)^{n-1}}.$$

   (d) A more general result than the latter bears Rolle's name today; state Rolle's Theorem.

   (e) State the Mean Value Theorem. *(Learn it, Love it, Live it!!)*

4. (a) Let $f(x) = |x| - 1$. Then $f(-1) = f(1) = 0$, but $f'(x) \neq 0$ on $[-1, 1]$. Does this contradict Rolle's Theorem? Explain!

   (b) Does the Mean Value Theorem apply to the function $f(x) = \frac{x^2-4x+3}{x-3}$ on $[2, 4]$?

   (c) Is there a point c on $[2, 4]$ for which $f'(c) = \frac{f(4)-f(2)}{4-2}$ where $f(x)$ is the function of part b)?

5. Prove that $\frac{1}{9} < \sqrt{66} - 8 < \frac{1}{8}$.

6. A quadratic function is a function of the form $f(x) = Ax^2 + Bx + C$ where $A \neq 0$.

   (a) Find a quadratic function satisfying: $f(3) = 0$, $f'(x) < 0$ if $x < 1$, and $f'(x) > 0$ if $x > 1$.

   (b) Find a quadratic function $f(x)$ satisfying: $f(0) = f'(0) = f''(0) = 2$.

   (c) Suppose that $f(x) = Ax^2 + Bx + C$ has roots r and s, *(i.e. f(r)=f(s)=0)*. Show that $f'(c) = 0$ where c is the midpoint between r and s.

   (d) Show that for any quadratic function $f(x) = Ax^2 + Bx + C$, the x-value c, guaranteed by the Mean Value Theorem applied to $f(x)$ on $[a, b]$ is always the midpoint between a and b.

7. (a) Suppose that an object lies at $x = 4$ when $t = 0$ and that the velocity $\frac{dx}{dt} = 35$ with a possible error of ± 1, for all $t \in [0, 2]$. Using the Mean Value Theorem what can you say about the object's position when $t = 2$.

   (b) The fuel comsumption of an automobile varies between 17 and 23 miles per gallon, according to the conditions of driving. Let $f(x)$ be the number of gallons of fuel left in the tank after x miles have been driven. If $f(100) = 15$, find upper and lower estimates for $f(200)$.

8. Prove the following assertions:

   (a) If $f'(x) > 0$ for all x, then $f(x)$ is increasing.

   (b) If $f'(x) = g'(x)$ for all x, then $f(x) = g(x) + c$.

ESP Math 408C *WS#15* *Fall 1993*

1. True or False. If false, give a counter-example. Also, for those terms in bold print which are false, give a correct definition.

   (a) $x = c$ is a **critical point** of $f(x)$ if $f(c) = 0$.

   (b) $f(x)$ has a **relative maximum** at $x = c$ if and only if $f'(c) = \frac{f(b)-f(a)}{b-a}$.

   (c) $f(x)$ has a **relative minimum** at $x = c$ if $f'(c) = 0$.

   (d) $f(x)$ is a function defined on $[a, b]$. There is a point $c \in (a, b)$ such that $f(x)$ is increasing for $x > c$ and decreasing for $x < c$. Thus, $f'(c) = 0$ and $f(c)$ is the minimum value of $f(x)$ on $[a, b]$.

2. Find the critical points, relative maxima and minima for the following.

$$a)\ f(x) = x^4 - 4x^3 \qquad\qquad b)\ f(x) = \sqrt{\frac{1+x^2}{2+x^2}}$$

3. Find the absolute extrema of the given functions on the indicated interval.

   (a) $f(x) = \cos 2x + 2\cos x$ on $0 \le x \le \pi$

   (b) $f(x) = 2\sin x - \cos 2x$ on $0 \le x \le 2\pi$

4. Show that the sum of a positive number and its reciprocal must be at least 2.

5. The mathematicians at Los Alamos Laboratory developed the following equation to describe the change over time in the number of people infected with the AIDS virus.

$$\frac{dI}{dt} = \alpha I(t)[1 - \frac{I(t)}{N}]$$

where

$$\begin{aligned} I(t) &= \text{ number of people infected at time } t \\ N &= \text{ size of the population} \\ \alpha &= \text{ rate at which an infected person passes on the virus per unit time.} \end{aligned}$$

Assume t is measured in days, $N = 100,000$, and $\alpha = 0.01$.

(a) Is the number of infected persons increasing or decreasing?

(b) How many people have to be infected for the number of people infected to stop increasing?

(c) How many people are being infected per day when there are 100 people infected?

(d) Find $\frac{d^2I}{dt^2}$ using implicit differentiation then write it as a function of $I(t)$.

(e) Is the number of infected people accelerating, or increasing at a slower and slower rate?

(f) How many people have to be infected for the number of people infected to stop accelerating?

1. A certain function $f(x)$ defined for all x has $f''(0) = 0$, $f''(x) > 0$ for $x > 0$, and $f''(x) < 0$ for $x < 0$. The number of critical points, number of local maxima, number of local minima, and number of roots of $f(x)$ are all tabulated. Give all possible such tabulations. *(**NOTE:** this problem appeared on Dr. Hamrick's 408C exam in 1988.)*

2. For the following functions:

   (a) Determine symmetry. Find roots and intercepts.

   (b) Take the derivative and simplify.

   (c) Find all the points where the derivative is zero or does not exist.

   (d) Using the first derivative test, find the intervals where the function is increasing and decreasing.

   (e) Find the relative extremes.

   (f) Find the second derivative and simplify.

   (g) Find where the second derivative is zero or does not exist.

   (h) Find the intervals where the function is concave up or down.

   (i) List the inflection points;

   (j) Plot points for the functions. Make sure you plot critical points, points of inflection, intercepts, and endpoints of the domain (if it is an interval).

   (k) Find the vertical and horizontal asymptotes (if they exist).

   (l) Sketch the graph.

a) $f(x) = x^4 + 4x$

b) $f(x) = \dfrac{x}{x^2+4}$

c) $f(x) = \dfrac{(1+x)\sqrt{1+x^2}}{x}$

d) $f(x) = \dfrac{x^2}{x^2-4}$

e) $f(x) = \dfrac{x^2}{\sqrt{x^2-2}}$

f) $f(x) = x^3 + \dfrac{3}{x}$

g) $f(x) = 2\cos x \sin x - x,$ on $[0, 2\pi]$

h) $2\tan x - \sec^2 x,$ on $(0, \dfrac{\pi}{2})$

3. (a) Show that $f(x) = x^5 + 10x^3 + 45x - 10$ has exactly one real root. Find the integer closest to it.

   (b) Show that $f(x) = x^5 - 10x^3 + 45x - 10$ has exactly one real root. Find the integer closest to it.

   (c) Find a seventh degree polynomial that has exactly one real root.

ESP Math 408C WS#17 Fall 1993

1. Early in 1492, Cristóbal Colón was commissioned by King Ferdinand and Queen Isabella of Spain to journey west to reach the Orient. On September 6, 1492, Columbus (Colón's name in Latin) left the Canary Islands to make history. Not knowing exactly which direction he should head, Columbus manages to get his ship moving east or west according to the function $x(t) = (t-36)^3\,(100-t)$. We pick up the action at $t = 0$ and follow him as $t \to \infty$. (Note that t is in days, $x(t)$ is in meters, and that we ignore the reality that Columbus sailed southwestward, not just east and west.)

   (a) Assuming September 6, 1492 is the day we start watching, what is the position of the Canary Islands?

   (b) Since we are given that Columbus is initially travelling west, what is your sign convention?

   (c) For which $t's$ is Columbus' ship stopped?

   (d) When is he moving west? east?

   (e) There is an island at $x = 0$. How many times does Columbus go past this island? When?

   (f) When did Columbus stop at this island, called Guanahani by its natives, which he claimed for Spain and renamed San Salvador? From this, can you calculate the date of this reknowned stop?

   (g) At $t = 67\frac{3}{4}$ days, Columbus realizes that his second-in-command, Pinzón, and the Pinta had vanished before a strong east wind. At this point, Columbus thought that it would be a good idea to go east to find his friend. When (for which time t) does the ship begin to slow down? What was the reaction time from seeing this to slowing down? How far did he travel this time?

   (h) For which t is he:

      i. speeding up in the westward direction?

      ii. slowing down in the westward direction?

      iii. speeding up in the eastward direction?

      iv. slowing down in the eastward direction?

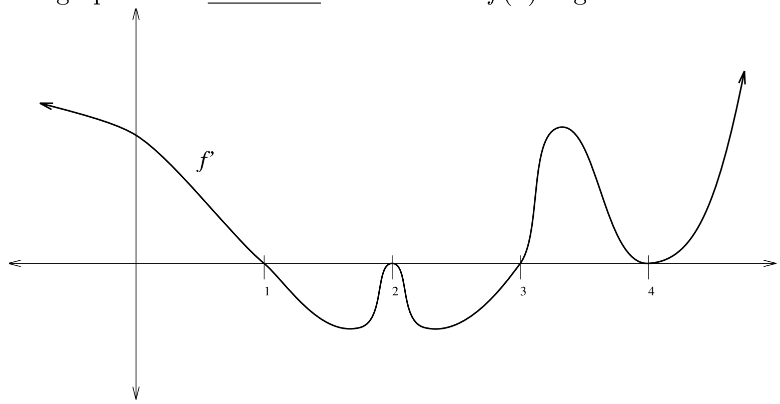
   (i) What is his position when he finally starts going east toward the island of Haiti which he named Española (Hispaniola in Latin)?

   (j) How many days elapse before Columbus sets his sights upon San Salvador once again?

   (k) Columbus is now headed back to Spain. At what time t on his way to Spain are we sure that Columbus has passed the Canary Islands?

   (l) How can we alter our function $x(t) = (t-36)^3\,(100-t)$ so that Columbus is allowed to stop at Española and board the Niña since the Santa Maria ran aground at Española and was a total loss? Furthermore, how can we ensure (by altering the function $x(t)$) that Columbus eventually stops as $t \to \infty$? Keep in mind that after his stop in Haiti, Columbus must travel east.

2. The graph of the derivative of a function $f(x)$ is given below.



(a) What are the critical points of $f(x)$?

(b) Which critical point(s) correspond to relative extrema?

(c) Can you determine any inflection points?

(d) Graph $f(x)$.

3. If $f(x) = ax^3 + bx^2 + cx + d$, find values for a,b,c, and d so that $f(x)$ has a local maximum at $x = -1$, $f(-1) = 2$; $f(x)$ has a local minimum at $x = 1$, $f(1) = -1$

1. Given $ax^2 + bx + c = 0$, derive the quadratic formula. *(Hint: you need to complete the square.)*

2. Prove that of all rectangles with given perimeter P, the square has the largest area.

3. Of all the triangles that pass through the point $(2, 1)$ and have two sides on the coordinate axes, find the dimensions of the one having the smallest area.

4. (a) Find the shortest line segment with endpoints on the x and y axes going through the point $(1, 8)$.
   (b) What is the area of the triangle formed by the shortest line segment?
   (c) What is the rate of change of area with respect to the $x - coordinate$ of the point on the $x - axis$?
   (d) For which x is the area increasing?

5. A garden is designed to be in the shape of a circular sector with radius R and angle θ. If the area A is to be a constant, find the dimensions (R, θ) which minimize the length of fence around the perimeter.

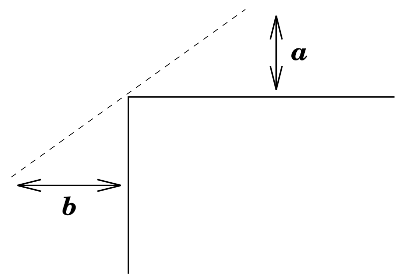
6. A sector with angle ϕ is cut from a circle of radius 12 inches, and the resulting edges are brought together to form a cone. Find the magnitude of ϕ so that the volume of the cone is maximum.

7. Twenty feet of wire are to be used to form two figures. In each of the following cases, how much should be used for each figure so that the total enclosed area is a maximum?

   (a) equilateral triangle and square
   (b) square and regular pentagon
   (c) regular pentagon and hexagon
   (d) regular hexagon and circle
   (e) What can you conclude from this pattern? *(Hint: The area of a regular polygon with n sides of length x is $A = (\frac{n}{4})(\cot \frac{\pi}{n})x^2$.)*

8. What are the dimensions of the lightest cylindrical aluminum can with capacity $1,000\ cm^3$.

9. Two hallways meet at right angles. Their widths are a and b as indicated in the picture. What is the greatest length of a ladder which can be crried horizontally around the corner?



10. A man has 100 meters of fencing and wishes to use it to enclose three equal and contiguous pens, using an existing building as a common side of the three pens. What are the dimensions of the pen arrangement that give the most total area inside the pens?

11. (a) Find the closest point on the graph of $f(x) = x^2$ to the point (a, b).

    (b) Show that the line connecting $(0, b)$ to the closest point is normal to the graph at that point.

12. (a) A rectangular poster is to have side margins of 2 inches, top and bottom margins of 4 inches, and 50 square inches of printed matter. What are the dimensions of the poster of least area that meets these specifications?

    (b) A sheet of paper contains 18 square feet. The top and bottom margins are 9 inches and the side margins are 6 inches. What are the dimensions of the page that has the largest printed area?

1. Sketch the graph of <u>one</u> function satisfying the following, indicate inflection points, relative maxima or minima.

   (a) $|f(x)| < 2$ for all x

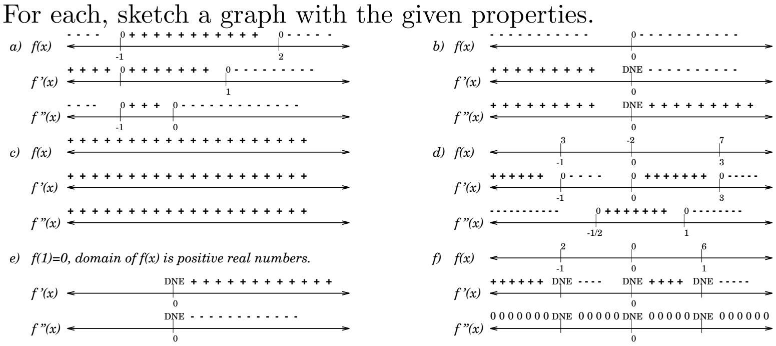
   (b) $f(-3) = f(-1) = 0$

   (c) $f'(x) < 0$ for $x < -2$, $f'(x) > 0$ for $x > -2$

   (d) $f''(x) < 0$ for $x < -3$ or $x > -1$

   (e) $f''(x) > 0$ for $-3 < x < -1$

2. For each, sketch a graph with the given properties.



3. Prove the following for a rational function $r(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

   (a) If $deg(p(x)) > deg(q(x))$ then $\lim_{x\to\infty} r(x) = \infty\ (or - \infty)$

   (b) If $deg(q(x)) > deg(p(x))$ then $\lim_{x\to\infty} r(x) = 0$

   (c) If $deg(p(x)) = deg(q(x))$ then $\lim_{x\to\infty} r(x) = \frac{\textit{leading coefficient of } p(x)}{\textit{leading coefficient of } q(x)}$

4. Find the following limits. Be sure to explain your reasoning at each step.

$$a)\ \lim_{x\to\infty} \frac{x + \sin x}{5x + 6} \qquad b)\ \lim_{x\to\infty} \frac{\sin x}{x} \qquad c)\ \lim_{x\to\infty} x \sin\frac{1}{x}$$

$$d)\ \lim_{x\to\infty} \frac{x \sin x}{x^2 + 5} \qquad e)\ \lim_{x\to\infty} \sqrt{x^2 + x} - x \qquad f)\ \lim_{x\to\infty} \frac{x^2(1 + \sin^2 x)}{(x + \sin x)^2}$$

5. Using the Mean Value Theorem, prove that if $f'(x) = 0$ for all x in an interval (a, b), then f is constant on the interval.

6. (a) Suppose that $f'(x) > g'(x)$ for all x and $f(a) = g(a)$. Show that $f(x) > g(x)$ for $x > a$ and $f(x) < g(x)$ for $x < a$.

   (b) Show by example that these conclusions do not follow without the hypothesis $f(a) = g(a)$.

7. A waste disposal corporation finds that if it invests i dollars to clean up a toxic waste site it will make a profit of

$$P(i) = 200i - 0.01i^2 \textit{ dollars.}$$

   (a) If it takes a minimum investment of \$12,500 to attempt the clean up, how much should they invest to maximize their profit? What if the minimum investment is \$5,000?

   (b) If it takes an investment of \$15,000 to get the job done right, how much should the company invest? What's the profit if they invest \$15,000?

   (c) The federal government decides to impose a fine of $F(i) = K - \frac{K}{15{,}000}\, i$ dollars for a job where i dollars is invested. For any value of K what is the fine if the job gets done right?

   (d) Suppose the minimum investment for attempting a job is \$5,000 and the fine function has $K = \$750,000$. How much should the company invest to maximize its profit? *(Taking in to account that it must pay a fine.)*

   (e) What should K be in order to insure that the maximum profit occurs when the job gets done right?

8. For each of the following functions find an integer n so that $f(x) = 0$ for some x between n and $n + 1$. *(Recall the Intermediate Value Theorem)*

a) $f(x) = x^3 - x + 3$	*b)* $f(x) = x^5 + x + 1$
b) $f(x) = 4x^2 - 4x + 1$	*d)* $f(x) = x^5 + 5x^4 + 2x + 1$

1. Fill in the derivatives.

   a) $\dfrac{d\sin x}{dx}$ *b)* $\dfrac{d\cos x}{dx}$ *c)* $\dfrac{d\tan x}{dx}$ *d)* $\dfrac{d\cot x}{dx}$ *e)* $\dfrac{d\sec x}{dx}$ *f)* $\dfrac{d\csc x}{dx}$

2. Fill in the antiderivatives which you know up to now.

   a) $\int \sin x\, dx$ *b)* $\int \cos x$ *c)* $\int \tan x$ *d)* $\int \cot x$ *e)* $\int \sec x$ *f)* $\int \csc x$

3. For each of the following solve the indefinite integral, describe your technique, then write two more integrals which would use the same technique to solve.

   a) $\displaystyle\int \frac{1}{\sqrt{2x+1}} \sin(\sqrt{2x+1})$ *b)* $\displaystyle\int (\sqrt{\tan x} + (\tan x)^{\frac{1}{3}}) \sec^2 x$

   c) $\displaystyle\int \cos^2 4x$ *d)* $\displaystyle\int x^{-\frac{2}{3}} \cos x^{\frac{1}{3}} (1 + \sin^2 x^{\frac{1}{3}})$

   e) $\displaystyle\int \frac{(\tan^{\frac{1}{3}} \sqrt{x})(1 + \tan^2 \sqrt{x})}{\sqrt{x}}$ *f)* $\displaystyle\int (1 - \cos^2 x)^{-1}$

4. Using the Mean Value Theorem, prove that if $f'(x) = 0$ for all x in an interval (a, b), then f is constant on the interval.

5. Prove that if F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$ *for all* x *in* I where C is a constant. *(Hint: Consider* $H(x) = G(x) - F(x)$ *and use the result from problem 4.)*

6. True or False. If False, make statement true (nontrivially).

   (a) If f is continuous on the closed interval $[a, b]$ then the area of the region bounded by the graph of f, the *x-axis*, and the vertical lines $x = a$ and $x = b$ is given by

   $$\textit{area} = \int_a^b f(x)$$

   (b) If a function f is continuous on the closed interval $[a, b]$ then f is integrable on $[a, b]$.

   (c) If f is defined at $x = a$, then $\int_a^a f(x) = a$

   (d) If f is integrable on $[a, b]$, then $\int_a^b f(x) = \int_b^a f(x)$.

7. Consider $f(x) = 2x - x^2$ on the interval $[0, 2]$.

   (a) Sketch the graph of $f(x)$.

   (b) Partition the interval $[0, 2]$ into subintervals of length $\frac{1}{2}$. Using this partition:

   i) Find a plausible way to compute an overestimate of the area bounded by $f(x)$, the $x - axis$, and the lines $x = 0$ and $x = 2$.

   ii) Find a plausible way to compute an underestimate of the area bounded by $f(x)$, the $x - axis$, and the lines $x = 0$ and $x = 2$.

   (c) Using the procedure you suggest above compute an overestimate and an underestimate of the area bounded by $f(x)$, the $x - axis$, and the lines $x = 0$ and $x = 2$.

   (d) Compute $\int_0^2 f(x)$. How does this compare to the estimates in c)?

   (e) Partition the interval $[0, 2]$ into subintervals of length $\frac{1}{4}$ and repeat parts c) and d) for this new partition. What do you notice? What conclusions can be made?

1. (a) State the Fundamental Theorem of Calculus.
   (b) State what your book calls the "Second Fundamental Theorem of Calculus."
   (c) State the Mean Value Theorem for integrals.
   (d) Using parts (a) and (b), prove part (c).
   (e) Why is (c) called the "Mean Value" Theorem for integrals?
   (f) Show graphically, if given a function $f(x)$ which is nonnegative and continuous on $[a, b]$, the areas which the Mean Value Theorem for integrals implies are equal.

2. True or False. If False, make statement true (nontrivially).

   (a) If $f(x)$ is continuous on the closed interval $[a, b]$ then the area of the region bounded by the graph of $f(x)$, the *x-axis*, and the vertical lines $x = a$ and $x = b$ is given by

   $$\textit{area} = \int_a^b f(x)$$

   (b) If a function $f(x)$ is continuous on the closed interval $[a, b]$ then $f(x)$ is integrable on $[a, b]$.
   (c) If $f(x)$ is defined at $x = a$, then $\int_a^a f(x) = a$
   (d) If f is integrable on $[a, b]$, then $\int_a^b f(x) = \int_b^a f(x)$.

3. (a) Prove that if $f(x)$ is integrable and nonnegative on the closed interval $[a, b]$ then $\int_a^b f(x) \geq 0$.
   (b) Use (a) to prove that if $f(x)$ and $g(x)$ are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)\ \forall\ x \in [a, b]$ then

   $$\int_a^b f(x) \leq \int_a^b g(x).$$

   (c) Show that

   $$-3 < \int_{-1}^{2} \frac{x^2 - 1}{x^2 + 1} < 2.$$

   (Hint: Use (a) or (b), find minimum and maximum values of the integrand on $[-1, 2]$*.)*
   (d) Show that

   $$\frac{1}{7\sqrt{2}} \leq \int_0^1 \frac{x^6}{\sqrt{1 + x^2}} \leq \frac{1}{7}.$$

4. (a) Calculate the derivative of $F()g(x)$, where F and g are continuously differentiable.

   (b) Let $F'(x) = f(x)$. Express $F()g(x) + C$, where C is constant, as the integral of its derivative.

   (c) Let $F(x) = \int_1^x f(t)\ dt$ and $G(x) = \int_2^x f(t)\ dt$. Show that F and G differ by a constant and find what the constant is. Draw a graph to see what F and G represent then compute $F(x) - G(x)$.

   (d) Compute the following integrals.

$$i)\ \int \sec(\sin x)\tan(\sin x)\cos x \qquad ii)\ \int \frac{\sqrt{2+\sqrt{x}}}{2\sqrt{x}}$$

5. (a) Find $H'(2)$ given that

$$H(x) = \int_0^{x^3-4} \frac{1}{1+\sqrt{t}}\ dt.$$

   (b) Find $H'(2)$ given that

$$H(x) = \int_{2x}^{x^3-4} \frac{1}{1+\sqrt{t}}\ dt.$$

   (c) Find $H'(3)$ given that

$$H(x) = \frac{1}{x}\int_3^x [2t - 3H'(t)]\ dt.$$

   (d) Find $F'(x)$ if $F(x) = \int_0^x xf(t)\ dt$.

   (e) Find all continuous functions satisfying $\int_0^x f(t)\ dt = [f(x)]^2 + C$. *(Hint: Differentiate both sides with respect to x.)*

6. Let $f(x)$ be differentiable on $[0,1]$, the second derivative exists for all $x \in [0,1]$ and $f'(0) = 0,\ f'(1) = 0,\ f(0) = 0,$ and $f(1) = 1$. Show that for some $a \in (0,1),\ |f''(a)| \geq 4$.

ESP Math 408C *WS#22* *Fall 1993*

1. The following integrals all require the same trick to solve using u-substitution. Solve the integrals and describe the trick.

$$a)\ \int x\sqrt{x+1} \qquad b)\ \int \frac{x+3}{\sqrt{x+1}} \qquad c)\ \int t^2(2t-1)^{-7}\,dt$$

$$d)\ \int t^3(t^2+6)^{\frac{1}{2}}\,dt \qquad e)\ \int x^8(x^3+1)^{-\frac{17}{2}}$$

2. Evaluate the integrals.

$$a)\ \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \csc x \cot x\,dx \qquad b)\ \int_0^1 (x-1)^{99}x\,dx \qquad c)\ \int_1^4 \frac{\sqrt{2+\sqrt{x}}}{\sqrt{x}}\,dx$$

$$d)\ \int_0^{\frac{\pi}{4}} \frac{\cos^2(\tan\theta)}{\cos^2\theta}\,d\theta \qquad e)\ \int_0^{2\pi} \sqrt{1-\sin^2 x}\,dx \qquad f)\ \int_2^3 \frac{\mathrm{dx}}{x^3-3x^2+3x-1}$$

$$g)\ \int \cos^4 x \qquad h)\ \int \tan^2 x \qquad i)\ \int \sin^5 x$$

3. Find the average value of the function on the interval, then find the "c" guaranteed by the Mean Value Theorem for Integrals. Graphically show the areas which the theorem implies are equal for part (b).

$$a)\ f(x) = \csc^2 x \text{ on } [\frac{\pi}{4}, \frac{3\pi}{4}] \qquad b)\ f(x) = x(x-1)^5 \text{ on } [1,2]$$

4. Calculate the following derivatives using the chain rule and the fundamental theorem of calculus.

$$a)\ \frac{ds}{dx}\int_0^x \frac{\mathrm{dt}}{1+t^2} \qquad b)\ \frac{ds}{dx}\int_0^{x^2} \frac{\mathrm{dt}}{1+t^2} \qquad c)\ \frac{ds}{dx}\int_{-x^2}^{x^2} \frac{\mathrm{dt}}{1+t^2}$$

$$d)\ \frac{d^2}{dx^2}[\int_0^x \frac{\mathrm{dt}}{1+t^2}] \qquad e)\ \frac{ds}{dx}\int_1^{\tan x} t^{10}\cos t\,dt \qquad f)\ \frac{ds}{dx}\int_{x^3}^{x^5+1} \frac{1}{t}\,dt$$

5. Suppose $f(x) = \frac{1}{2\sqrt{x+1}}$. Evaluate the following definite integrals.

$$a)\ \int_0^3 f'(x)\,dx \qquad b)\ \int_0^3 f(x)\,dx \qquad c)\ \int d(\int_0^x f'(t)\,dt)03x$$

6. We are given a differentiable, odd function $f(x)$ defined on $[-3,3]$ which has zeros at $x = -2, 0,$ and 2 (nowhere else) and critical points at $x = -1$ and $x = 1$ (nowhere else). Also, we know that $f(-1) = 1$. Define a new function F on $[-3,3]$ by the formula

$$F(x) = \int_{-2}^{x} f(t)\, dt$$

   (a) Sketch a rough graph of $f(x)$.

   (b) Find the value of $F(-2)$, $F(2)$, and an upper and lower bound on $F(0)$.

   (c) Find the critical points and inflection points of $F(x)$ on $[-3,3]$.

   (d) Sketch a rough graph of $F(x)$ on $[-3,3]$.

   (e) Interpret the points found in (c) in terms of the graphs of $f(x)$ and $F(x)$.

7. In each of the following sketch a graph(s) with the properties:

   (a) $f(x)$ is not constant 0 and $\int_{-a}^{a} f(x)\, dx = 0$.

   (b) $f(x) < 0$ for $x \in (-2,-1)$ and $\int_{-3}^{0} f(x)\, dx > 0$.

   (c) $f(x),\ g(x) < 0\ \forall\ x$ and $\int_{1}^{2} (f(x) - g(x))\, dx > 0$.

   (d) $\int_{0}^{2\pi} f(x)\, dx = 0,\ \int_{0}^{2\pi} |f(x)|\, dx = 2\int_{0}^{\pi} f(x)\, dx$.

8. (a) The region under the graph of $y = -2x+4$ on $[-2,1]$ is to be divided into two parts of equal area by a vertical line. Where should the line be drawn?

   (b) Where would you draw a horizontal line to divide the region in part (a) into two parts of equal area?

1. Evaluate the integrals.

$$a) \int_0^1 \frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx \qquad b) \int_0^{\sqrt{\frac{\pi}{2}}} \frac{\tan(\sec\sqrt{x})\sec\sqrt{x}\tan\sqrt{x}}{\sqrt{x}\cos^2(\frac{1}{\cos\sqrt{x}})}$$

$$c) \int_{-2}^2 |x^4-4x^2+3|\,dx \qquad d) \int_0^{2\pi} \sqrt{1-\cos^2\theta}\,d\theta$$

2. Let $F(x) = \int_1^x \frac{dt}{t}$, $x > 0$. For the following <u>do not use</u> the fact that you know another name for this function.

   (a) Find $F'(x)$ and determine intervals where it is increasing and decreasing.

   (b) Evaluate the following limits, give $+\infty$ or $-\infty$ for the limit when appropriate (i.e. instead of just DNE).

   $$\lim_{x\to\infty} F(x), \qquad \lim_{x\to 0^+} F(x), \qquad \lim_{x\to 0^+} \frac{1}{x}, \qquad \lim_{x\to\infty} \frac{1}{x}.$$

   (c) Find $F''(x)$ and determine the concavity of $F(x)$.

   (d) Graph $F(x)$.

   (e) Estimate $F(2)$ using differentials. Is it an overestimate or an underestimate? Why?

   (f) Just in case you haven't been reading ahead—the natural logarithm function is defined as
   $\ln x = \int_1^x \frac{dt}{t}$, $x > 0$.

3. Evaluate the following integrals.

$$a) \int \sin x \qquad b) \int \cos x \qquad c) \int \tan x \qquad d) \int \cot x \qquad e) \int \sec x \qquad f) \int \csc x$$

   (Hint for (e) multilply by $\frac{\sec x+\tan x}{\sec x+\tan x}$*.)*

4. Prove that $\int_1^a \frac{dt}{t} = \int_b^{ab} \frac{dt}{t}$. Use it to prove that $\int_1^a \frac{dt}{t} + \int_1^b \frac{dt}{t} = \int_1^{ab} \frac{dt}{t}$. What does this statement give if written in "ln" notation?

5. Use differentials to show that for small x, $\ln(1+x) \approx x$.

6. (a) Prove that $0 \leq \int_0^2 x^3 \, dx \leq 16$.

   (b) Use the trapezoidal rule with $n = 4$ to find an approximate value of $\int_0^2 x^3 \, dx$. Compare the result with the exact value.

   (c) Calculate $\int_{\frac{3}{5}}^{1} \sqrt{1 - x^2} \, dx$ by the trapezoidal rule and by Simpson's rule, both with $\Delta x = \frac{1}{5}$. Do not use a calculator.

7. Prove that Simpson's rule is exact when approximating the integral of a cubic polynomial function, and demonstrate the result for

$$\int_0^1 (x^3 - 3x^2 + 3x - 1) \, dx, \quad n = 2.$$

1. Solve the following differential equations subject to the prescribed initial conditions.

a) $\dfrac{dy}{dx} = 4(x-7)^3 \quad x = 8,\ y = 10$ b) $\dfrac{dy}{dx} = \dfrac{x^2+1}{x^2} \quad x = 1,\ y = 1$

c) $\dfrac{dy}{dx} = x\sqrt{y} \quad x = 0,\ y = 1$ b) $\dfrac{dy}{dx} = \dfrac{4\sqrt{(1+y^2)^3}}{y} \quad x = 0,\ y = 1$

2. Find all continuous functions $f(x)$ satisfying

$$\int_0^x f(t)\,dt = [f(x)]^2 + C$$

(Hint: Differentiate both sides with respect to x.)

3. (a) Consider the points on the graph of $y = x^2$ whose y coordinates are y_0. How far are these points from the y-axis?
   (b) Find the area bounded by $y = x^2$ and the line $y = 1$ by integrating along the y-axis.
   (c) Find the area of the triangle formed by the line $y = -2x + 4$ and the x and y axes in two ways: first integrate alone the x-axis then along the y-axis.
   (d) Find the area bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$ in two ways (as in (c)). Which method is easier?

4. Sketch and set up the integral for the area between the curves.

a) $y = x^2,\ \ y = 12 - x$ b) $y = \sin x - 1,\ \ y = -\dfrac{1}{2},\ \ x = 0,\ \ x = \dfrac{3\pi}{2}$

c) $y^2 = x,\ \ y^2 = 2 - x$ d) $y = x(4-x),\ \ y = (x-2)(x-4)$

5. (a) Rewrite the area in 4(a) as an integral with respect to y.
   (b) Calculate the area in 4(b).
   (c) Calculate the area in 4(d).
   (d) Rewrite the area in 4(c) as an integral with respect to the “other” variable.

6. For each of the following functions:

   (a) Find a positive integer n so that $f(x) = 0$ for some x between n and $n + 1$.

   (b) What would your choice of a *reasonable* estimate for a positive root of $f(x) = 0$ be?

   (c) Use Newton's method to approximate the root of $f(x) = 0$ within 0.01.

   i) $f(x) = x^3 - 2x + 1$ *ii)* $f(x) = x^2 - 4x + 1$ *iii)* $f(x) = 4x^3 + x - 112$

1. Find the domain and range of the following functions.

   a) $f(x) = \ln x^2$ *b)* $f(x) = 2\ \ln x$ *c)* $f(x) = \ln|\sin x|$ *d)* $f(x) = \ln(\sin x)$

   Now, explain why we must be careful when integration involves logarithmic functions.

2. Evaluate the following integrals.

   a) $\int \sin x$ *b)* $\int \cos x$ *c)* $\int \tan x$ *d)* $\int \cot x$ *e)* $\int \sec x$ *f)* $\int \csc x$

   (Hint for (e) multilplply by $\frac{\sec x + \tan x}{\sec x + \tan x}$*.)*

3. Let $f(x) = \frac{\ln x}{x}$ and $x > 0$.

   (a) Sketch the graph of $f(x)$, indicate relative extremes and inflection points.

   (b) Find the equation of the tangent line to the graph which passes through the origin.

   (c) For very large x, which is bigger $\ln x$ or x? Use your graph, not the calculator.

   (d) Use differentials to show that for small x, $\ln(1 + x) \approx x$.

4. Let $f(x)$ be a differentiable function and $g(x) = f^{-1}(x)$. Given the latter, recall that $f(g(x)) = x$. Prove that

$$g'(x) = \frac{1}{f'(g(x))}.$$

5. Suppose the function g is the inverse of the function f. Show the plausibility of the following statements in terms of the graphs, and then give a one line proof of each.

   (a) If f is decreasing at a point, then g is decreasing at the point.

   (b) If f is decreasing at a point, then f and g are either both concave up or both concave down at the point.

   (c) If f is increasing at a point, then g is also increasing at the point.

   (d) If f is increasing at a point, then f and g have opposite concavity at the point.

6. Suppose that g and h are increasing functions on an interval I. For the following functions, either show that they must be increasing on I or give a counter-example.

   a) $g + h$ *b)* $g \cdot h$ *c)* $g \circ h$

7. Let $f(x) = 2x + \log_e x$ for $x > 0$.

   (a) Explain why f has an inverse function.

   (b) For what values of x is $f^{-1}(x)$ defined?

   (c) Find $g'(2)$, where $g(x) = f^{-1}(x)$.

8. In 1931, Volterra obtained differential equations of the forms

$$\frac{dx}{dt} = x(a_1 + a_2 y) \text{ and } \frac{dy}{dt} = y(b_1 + b_2 x)$$

as a mathematical model describing the competition between two species coexisting in a given environment. From this system one obtains (by the chain rule) the differential equation

$$\frac{dy}{dx} = \frac{y(b_1 + b_2 x)}{x(a_1 + a_2 y)}$$

Solve this differential equation.

ESP Math 408C WS#26 Fall 1993

1. True or false? For those which are true only if the domain is restricted, give the domain.

a) $\log_b xy = (\log_b x)(\log_b y)$ *b)* $\log_b (x+y) = \log_b x + \log_b y$

c) $\log_b (\frac{x}{y}) = \log_b x + \log_b y$ *d)* $\frac{1}{2} \log_b x = \log_b \sqrt{x}$

e) $\log_{10} e = (\ln 10)^{-1}$ *f)* $(\log_b x)^r = r \log_b x$

g) $\log_b x = (\int_1^x \frac{dt}{t})(\int_1^b \frac{dt}{t})^{-1}$ *h)* $y = b^x$ iff $\log_b y = x$

i) $b^{\log_b (Texas)} = Texas$

2. Solve for x.

a) $x^{10} = 1000$ *b)* $3 \log_x 4 = 2$ *c)* $3^x = 4^{2x-1}$

d) $\log_x (1-x) = 2$ *e)* $\log_2 x = \log_4 5 + 3 \log_2 3$ *f)* $\ln x + \ln (x-2) = 0$

3. Given that you know $\frac{ds}{dx} e^u = e^u \frac{du}{dx}$, prove the following.

a) $\frac{ds}{dx} a^x = a^x \ln a$ *b)* $\frac{ds}{dx} a^u = a^u \ln a \frac{du}{dx}$

c) $\frac{ds}{dx} \log_a x = \frac{1}{x \ln a}$ *d)* $\frac{ds}{dx} \log_a u = \frac{1}{u \ln a} \frac{du}{dx}$

(Hint: rewrite a^x in terms of the exponential function, then differentiate)

4. (a) Show that $(\log_e \pi)(\log_\pi e) = 1$. Generalize.
   (b) By analyzing the function $f(x) = \frac{\ln x}{x}$, show that $\pi^e < e^\pi$. Generalize.

5. Sketch the graphs of the following functions and give their domains.

a) $f(x) = \log_{\frac{1}{2}} x$ *b)* $f(x) = \frac{1}{2}^x$ *c)* $f(x) = 10^{\sqrt{1-x^2}}$

6. (a) For what values of b is $f(x) = \log_b x$ an increasing function?
   (b) Sketch the functions $\log_e x$ and $\log_{\frac{1}{e}} x$.
   (c) For what values of b is $f(x) = b^x$ and increasing function?
   (d) Sketch the functions e^x and $(\frac{1}{e})^x$.

7. Differentiate.

a) $f(x) = x^{\cos x}$ *b)* $f(x) = (\cos x)^x$ *c)* $f(x) = 10^{e^{x^2}}$

d) $f(x) = (x^x)^x$ *e)* $f(x) = x^{x^x}$

8. Integrate the following.

a) $\int x\, e^{x^2}$ *b*) $\int \cos x\, e^{\sin x}$ *c*) $\int \sqrt{10^{3x}}$

d) $\int f'(x)\, a^{f(x)}$ *e*) $\int (\cos x)^x (\ln\cos x - x\ \tan x)$ *f*) $\int 2xe^{x^2}(10)^{2e^{x^2}}$

1. (a) Which of the following is true? If f is integrable on $[a, b]$ then

$$i)\ |\int_a^b f(x)| \le \int_a^b |f(x)| \qquad ii)\ \int_a^b |f(x)| \le |\int_a^b f(x)|$$

(b) Prove the one in (a) that is true and give a counter-example for the one in (a) that is false.

(c) Using the above, prove that

$$i)\ |\int_0^1 \frac{\cos nx}{x+1}| \le \ln 2 \text{ for all n} \qquad ii)\ |\int_0^{e-1} \frac{\sin nx}{x+1}| \le 1 \text{ for all n}$$

2. Assume that a population grows according to Verhulst's logistic law of population growth

$$\frac{dN}{dt} = AN - BN^2,$$

where $N = N(t)$ is the population at time t, and the constants A and B are the "vital coefficients" of the population. What will the size of this population be at any time t? What will the population be after a very long time, that is, as $t \to \infty$. Why do you think A and B are called "vital coeficients"? [Hint: $\frac{1}{AN-BN^2} = \frac{-1}{AN} + \frac{B}{A(BN-A)}$]

3. Suppose that on some interval the function f satisfies $f'(x) = cf(x)$ for some number c.

(a) Assuming that f is never zero, prove that $|f(x)| = Ae^{cx}$ for some number A ($A > 0$). It follows that $f(x) = Ke^{cx}$ for some K.

(b) Show that the result holds without the added assumption that f is never 0. *(Hint: Show that f can't be 0 at the endpoint of an open interval on which it is nowhere 0.)*

(c) Give a simpler proof that $f(x) = Ke^{cx}$ for some K by considering the function $g(x) = \frac{f(x)}{e^{cx}}$.

(d) Suppose that $f'(x) = f(x)g'(x)$ for some g. Show that $f(x) = Ke^{g(x)}$ for some K.

4. Assume that the rate at which a radioactive substance decays is proportional to the amount present. In a certain sample 50% of the substance dissappears in a period of 1600 years. (This is true, for example, for radium 226.)

   (a) Write the differential equation that describes the decay of the substance.

   (b) Compute the decay constant of the substance.

   (c) What percentage of the original sample will disappear in 800 years?

   (d) In how many years will only one fifth of the original amount remain?

5. James' definition of the $\frac{1}{n}th$-life of a radioactive material which decays continuously at a rate proportional to the amount present is the number of years required for $\frac{n-1}{n}$ of the atoms in sample of radioactive material to decay.

   (a) Find a general formula for the $\frac{1}{n}th$-life. (i.e. Use $P(t) = P_0 e^{kt}$ and other information to get a formula for the $\frac{1}{n}th$-life, $t_{\frac{1}{n}}$.)

   (b) Suppose that after one year, only 36.79% of an initial amount of radioactive material as above remains. Find the 13-life.

   (c) For (b) find the 35-life.

   (d) From the information given, can you compute the initial amount present in part (b)?

   (e) Suppose that after one year, we also know that we have 2.7 grams of material left. Now, give the original amount present.

   (f) The $\frac{1}{n}th$-life, $t_{\frac{1}{n}}$, satisfies the following equation: $P(t + t_{\frac{1}{n}}) = \frac{1}{n}P(t)$. Why?

6. According to Newton's law of cooling, the rate at which the temperature of an object changes is proportional to the difference between its temperature and that of its surroundings. A cup of coffee at 200^o in a room of temperature 70^o is stirred continually and reaches 100^o after 10 minutes. At what time was it at 120^o?

1. Derive the formulas below. *(Hint: differentiate $g(g^{-1}(x)) = x$ then use triangles (e.g. you know that* $\arcsin x$ *is an angle, θ, whose sine is x.).)*

$$a)\ \frac{ds}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}} \qquad b)\ \frac{ds}{dx}\arctan x = \frac{1}{1+x^2}$$

2. Evaluate the given expression without a calculator. *(Hint: Use right triangles.)*

$$a)\ \sec(\arcsin\frac{4}{5}) \qquad b)\ \csc[\arctan(-\frac{5}{12})]$$

3. Differentiate each of the following functions.

$$a)\ f(x) = \arctan(\arctan(\arctan x)) \qquad b)\ f(x) = \arcsin(\arctan(\arccos x))$$

4. (a) Let $f(x) = \arcsin x + \arccos x$. Give two separate arguments to show that f is constant. What is that constant?

   (b) Let $f(x) = \arcsin(\cos x)$, $0 \le x \le \pi$. Show that $f(x) = ax + b$ for constants a and b, find them.

5. Let

$$F = \int_0^x \frac{dt}{1+t^2} + \int_0^{\frac{1}{x}} \frac{dt}{1+t^2}.$$

Show by two separate arguments that F is constant on $(-\infty, 0)$ and constant on $(0, \infty)$.

6. Compute the derivatives of the following functions.

$$a)\ y = [\ln(\cos x)]e^{\arcsin x} \qquad b)\ y = \ln(\sqrt{\frac{x+1}{x-1}}) \qquad c)\ y = x^{x^2}$$

7. Evaluate the integral. (Don't forget the constant!)

$$a)\ \int \frac{x^2-1}{x^2+1} \qquad b)\ \int \frac{x+1}{x+2} \qquad c)\ \int \frac{x^3+1}{x^2+x+1}$$

8. Find all continuous functions $f(x)$ which satisfy the equation

$$(f(x))^2 = \int_0^x f(t)\frac{t}{1+t^2}\,dt.$$

1. *Newton's Law of Cooling* states that the rate of change of the temperature T of an object is proportional to the difference between T and the temperature τ of the surrounding medium:

$$\frac{dT}{dt} = k(T - \tau)$$

where k is a constant.

   (a) Solve this equation for T. (the general solution)

   (b) Solve this equation for T given that $T(0) = T_0$.

   (c) Take $T_0 > \tau$. What is the limiting temperature to which the object cools as t increases? What happens if $T_0 < \tau$?

2. A cup of coffee is served to you at 185^o F in a room where the temperature is 65^o F. Two minutes later the temperature of the coffee has dropped to 155^o F. How many more minutes would you expect to wait for the coffee to cool to 105^o F?

3. How long does it take for a sum of money to double when compounded continuously (a) at 6%. (b) at 8%. (c) at 10%.

4. At what rate r of continuous compounding does a sum of money triple in 20 years?

5. A year ago there were 4 grams of a radioactive substance. Now there are 3 grams. How much was there 10 years ago?

6. (a) Let $f(x) = x\ln(\frac{1}{x})$. In what intervals does $f(x)$ have a well defined inverse?

   (b) Let $f(x) = 2x + \ln x$. Note that $f(1) = 2$ and $f^{-1}(x)$ is well defined in an interval about 2. What is $\frac{ds}{dx}f^{-1}(2)$?

7. Differentiate the following. Use logarithmic differentiation where necessary.

a) $f(x) = e^{e^{e^x}}$	*b)* $f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x}$
c) $f(x) = \log_7(e^x)$	*d)* $f(x) = e^{(\int_0^x e^{-t^2}\,dt)}$
e) $f(x) = \ln(1 + \ln(1 + x))$	*f)* $f(x) = \log_{(e^x)}(\frac{x}{\sin x})$

8. Integrate.

$$a) \int \frac{x}{2x+1} \qquad b) \int \frac{1+e^{2x}}{e^x} \qquad c) \int 2^x 3^x$$
$$d) \int x(e^{-x^2}+2) \qquad e) \int \frac{dx}{\sqrt{x}e^{\sqrt{x}}} \qquad f) \int 5^{\log 3} 7^{\sin x} \cos x$$

9. Find the constant k so that the graphs of $y = e^{4x}$ and $y = kx^2$ are tangent to one another at some point.

10. On December 31, Jaime gets five numbers correct in the Texas lottery and wins \$900. He decides to save a portion of it because he knows his tuition bill for the next fall will be at least \$904.67. He is lucky enough to find a bank which will continuously compound his investment at a rate of 6%. The tuition is due by August 1st. How much must he save in order to have at least \$904.67 when the tuition is due? How much will he have left over to share with the rest of the Emerging Scholars, including the instructor and the student assistants? More realistically, given today's interest rates and banks, how much should he save at a rate of 3.5% compounded quarterly? Would he have enough left over to throw a New Year's Party which will cost him at most \$150? What rate compounded monthly would be necessary in order for Jaime to save the required amount, but spend \$500 on a special trip?

1. The density (weight per unit length) of a straight wire of length L varies along its length according to the formula

$$\delta(x) = kx + b \ \frac{\text{grams}}{\text{cm}} \quad (0 < x < L)$$

where k and b are positive constants.

 (a) Draw a graph of $\delta(x)$ and interpret the *weight* of the wire in terms of some feature of this graph.
 (b) Write a definite integral that gives the weight of the wire.
 (c) Evaluate the integral using either calculus or geometry.

2. Sketch the region bounded by the curves. Represent the area of the region as one or more integrals in terms of x.

$$a)\ y = \arcsin x, \quad y = \frac{\pi}{2}, \quad x = 0$$
$$b)\ y = \frac{2x+1}{\sqrt{x^2+x+1}}, \quad y = -\frac{1}{\sqrt{x^2+x+1}}, \quad x = \frac{3}{2}$$
$$c)\ x = |y|, \quad y^2 = 2 - x,$$
$$d)\ 8x = y^3, \quad 8x = 2y^3 + y^2 - 2y$$

3. (a) Rewrite the area in 2(c) as an integral with respect to y.
 (b) Calculate the area in 2(d).
 (c) Calculate the area in 2(b).
 (d) Rewrite the area in 2(a) as an integral with respect to y. Which integral can you calculate?

4. The graphs of $y = x^4 - 2x^2 + 1$ and $y = 1 - x^2$ intersect at three points. However, the area between the curves *can* be found by a single integral. Explain why this is so, and write an integral for this area.

5. Evaluate the following integrals.

$$a)\ \int \frac{dx}{7+3x^2} \qquad b)\ \int \frac{x}{\sqrt{1-4x^4}} \qquad c)\ \int \frac{x+4}{x^2+1} \qquad d)\ \int \frac{3x^2+3x+1}{x^3+2x^2+2x+1}$$

6. Show that

$$\frac{1}{9} < \int_{-\sqrt{2}}^{\sqrt{2}} e^{-x^2}\, dx < 3.$$

What could you do to sharpen the upper and lower bounds of this integral? Sharpen them non-trivially.

1. (a) Let Ω be the region bounded by the curves $y = 3 - x^2$, $y = 2x$, and $x = 0$. Sketch the graphs labeling the point of intersection in the first quadrant.
   (b) Find the area of Ω.
   (c) Using discs find the volume of the solid obtained by revolving Ω about the x-axis.
   (d) Using shells find the volume of the solid obtained by revolving Ω about the y-axis.

2. Find the volumes of the following solids using calculus.
   (a) A ball of radius r.
   (b) A right circular cone of radius r and height h.
   (c) A right circular cylinder of height h and radius r.

3. Find the volume of the following solids obtained by rotating:
   (a) the region bounded by $y = \frac{1}{x}$, $x = 1$, $x = y$, $y = 0$ around the y-axis;
   (b) the region bounded by $y = \arctan x$, $x = 0$, $y = \frac{\pi}{4}$ around the y-axis;
   (c) the region bounded by $x = y^2$, $y = \frac{x}{2}$ around the line $x = 2$.

4. Derive the volumes for the following solids.
   (a) A right pyramid whose altitude is h and whose base is a square with sides of length a.
   (b) The water which is two inches deep in a hemispherical basin of radius one foot.
   (c) A solid object with cross sections being squares of side length $s(x) = \sqrt{\sin x}$ $0 \leq x \leq \pi$.
   (d) The intersection of two tubes of radius r at right angles.

5. Since El Paso is such a long way from Austin, Tasha and Olivia decide to make a little money for a flight home by helping out a local painter. The painter had always been fond of the curve $y = \frac{1}{x}$ and wanted them to engineer a paint can utilizing his favorite curve. They decide create a paint can by revolving the curve $y = \frac{1}{x}$ from $x = 1$ to $x = H$ about the x-axis.

   (a) What is the *volume* of paint the can holds?

(b) What is the *area* of the paper strip between $y = \frac{1}{x}$, $y = -\frac{1}{x}$, and $x = 1$, $x = H$?

(c) Now, by letting $H \to \infty$ in your answers to (a) and (b) give the volume and area of the infinitely long paint can, and paper strip.

(d) If you dip the infinitely long paper strip into the full paint can, will there be enough paint to cover it?

1. Find the volume of the solid formed by revolving the region bounded by $y = e^x,\ y = 0,\ x = 0,$ and $x = 1$ about the x-axis using: (a) disc method; (b) shell method.

2. Find the arclengths of the graphs:

$$a)\ f(x) = \frac{2}{3}(x-7)^{\frac{3}{2}} \text{ on } [7,14]; \qquad b)\ f(x) = \frac{2}{3}(x-6)^{\frac{3}{2}} \text{ on } [6,12];$$
$$c)\ f(x) = \frac{2}{3}x^{\frac{2}{3}} \text{ on } [0,3]; \qquad d)\ f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x \text{ on } [1,2].$$

3. A right circular cone is generated by revolving the region bounded by $y = \frac{hx}{r},\ y = h,$ and $x = 0$ about the y-axis. Verify that the lateral surface area of the cone is $S = \pi r\sqrt{r^2 + h^2}$.

4. A sphere of radius r is generated by revolving the graph of $y = \sqrt{r^2 - x^2}$ about the x-axis. Verify that the surface area of the sphere is $4\pi r^2$.

5. For the following, *set up* the integrals which represent the volume using the shell method *and* disc method (i.e. set up the integrals two ways). Decide which integral would be easier to calculate (but don't do it!).

   (a) the region bounded by $y = x^3 - 3x^2 + 2x,\ x = 0,\ x = 1,\ y = 0$ rotated around the y-axis;

   (b) the region bounded by $y = \cos(x^2),\ x = 0,\ x = \sqrt{\pi},\ y = 0$ rotated around the y-axis;

   (c) the region bounded by $y = \cos x,\ y = \sin x,\ x = 0,\ x = \frac{\pi}{4}$ rotated about the x-axis;

   (d) the region bounded by $9 - x = (y-3)^2,\ x = 0$ rotated about the x-axis.

6. Find the volume left over after a sphere of radius R has a hole of radius $\frac{1}{2}R$ drilled through the center.

7. Marcellus and Albert were working out at the Geometrically-Ideal Super Gym where all the dumbbells were constructed according to the model shown below.

   (a) Find the volume of the dumbbell pictured above.

   (b) Suppose that they needed a dumbbell suitable for bench pressing 200 pounds and $h = 4$ feet, what should the density of the material used to make the dumbbell be?

   (c) Ramiro joins the group and wants to sketch funny pictures on the dumbbells, how much drawing space does he have on the dumbbell shown above?

8. Show that the arc length integral gives the expected result for a linear function

$$f(x) = mx + b, \quad a \leq x \leq c.$$

1. Consider $g(x) = u(x)v(x)$ where $u(x)$ and $v(x)$ are differentiable with respect to x.

   (a) Find $\frac{dg(x)}{dx}$.

   (b) Find an expression for $\int u\,dv$ using part (a).

   $$(Notice: \qquad \int u\,dv = \int u(x)\frac{dv(x)}{dx}\,dx = \int u(x)\frac{dv(x)}{dx}dx)$$

   (c) Use (b) to compute the following integrals.

   $$i)\ \int \frac{1+\sin x}{\cos^2 x} \qquad ii)\ \int \cos^3(\frac{x}{2})\sin x \qquad iii)\ \int \frac{dx}{1+e^x}$$

   $$iv)\ \int x\sin x \qquad v)\ \int x^2 \ln x \qquad vi)\ \int x^3 e^{2x}$$

2. In some cases, integration by parts can be used when there is only one "part". Evaluate these two integrals.

   $$a)\ \int \operatorname{arcsec} x \qquad b)\ \int \ln x$$

3. An integral can sometimes be evaluated by carrying out integration by parts twice:

   $$\int e^{ax} \sin bx$$

4. Find the volume of the "torus" (shown below) obtained by rotating the circle $(x-a)^2 + y^2 = b^2$ $(a > b)$ around the y-axis.

5. A damping force affects the vibration of a spring so that the displacement of the spring is given by: $y = e^{-4t}(\cos 2t + 5\sin 2t)$. Find the average value of y on the interval from $t = 0$ to $t = \pi$.

6. Evaluate:

$$a)\ \int \frac{dx}{1+\sin x} \qquad b)\ \int e^{\sqrt{x}} \qquad c)\ \int x^x \ln(e\cdot x)$$

$$d)\ \int \frac{1}{\sqrt{x}\sqrt{1-x}} \qquad e)\ \int x^2 e^{2x^3} \qquad f)\ \int \frac{x^2+2x+1}{x^2+1}$$

$$g)\ \int \frac{2^{\ln x}}{x} \qquad h)\ \int \frac{x}{1+\sin x} \qquad i)\ \int (\ln x)^3$$

$$j)\ \int \frac{\ln(\ln x)}{x} \qquad k)\ \int \sec^3 x \qquad l)\ \int \cos(\ln x)$$

7. Find the fallacy in the following argument that $0 = 1$.

$$\begin{array}{rcc}
dv = dx & \Rightarrow & v = x \\
u = \dfrac{1}{x} & \Rightarrow & du = -\dfrac{1}{x^2}dx \\
0 + \displaystyle\int \frac{dx}{x} & = & (\dfrac{1}{x})x - \displaystyle\int (\frac{-1}{x^2})x \\
 & = & 1 + \displaystyle\int \frac{dx}{x} \\
0 & = & 1
\end{array}$$

8. Find the volume of the solid formed by revolving the region bounded by $y = e^x,\ y = 0,\ x = 0,$ and $x = 1$ about the y-axis using: (a) disc method; (b) shell method.

1. These integrals are useful in trigonometric substitution. You should remember them or be able to evaluate them quickly.

$$a)\ \int \sin^2\theta \qquad b)\ \int \cos^2\theta \qquad c)\ \int \sec^2\theta \qquad d)\ \int \tan^2\theta$$

$$e)\ \int \sec\theta \qquad f)\ \int \csc\theta \qquad g)\ \int \tan\theta \qquad h)\ \int \cot\theta$$

2. Evaluate.

$$a)\ \int \frac{\sqrt{9x^2-4}}{x} \qquad b)\ \int \frac{dx}{x^2\sqrt{9-x^2}} \qquad c)\ \int \frac{dx}{\sqrt{4x^2+9}}$$

$$(let\ \ x = \frac{2}{3}\sec u) \qquad (let\ \ x = 3\,\sin u) \qquad (let\ \ x = \frac{3}{2}\tan u)$$

3. Evaluate each of the following integrals using the substitution $x = \sin\theta$. In the last three problems, check by using another method.

$$a)\ \int \frac{dx}{\sqrt{1-x^2}} \qquad b)\ \int \sqrt{1-x^2} \qquad c)\ \int \frac{dx}{(1-x^2)^{\frac{3}{2}}}$$

$$d)\ \int (1-x^2) \qquad e)\ \int \frac{x}{\sqrt{1-x^2}} \qquad f)\ \int x(1-x^2)^{\frac{3}{2}}$$

4. Evaluate.

$$a)\ \int \sin^2\theta\cos^2\theta \qquad b)\ \int \sin^3\theta\cos^2\theta \qquad c)\ \int l\sin^2 2\,\theta\frac{\pi}{8}\frac{\pi}{4}$$

$$d)\ \int l\tan^3\theta 0\frac{\pi}{4} \qquad e)\ \int \sec^5\theta\tan^3\theta \qquad f)\ \int \frac{e^{2x}}{\sqrt{e^x+a}}$$

$$g)\ \int \sin\theta(6+\sec^2\theta) \qquad h)\ \int 30\tan^6\theta \qquad i)\ \int e^{\sin\theta}(\frac{\theta\cos^3\theta-\sin\theta}{\cos^2\theta})$$

5. Suppose that f'' is continuous and that

$$\int_0^{\pi} [f(x) + f''(x)]\,\sin x = 2.$$

Given that $f(\pi) = 1$, compute $f(0)$.

6. (a) Find $\int \arcsin x$. What "trick" did you use which is similar to the one used when integrating $\ln x$.

   (b) Generalize this trick: Find $\int f^{-1}(x)$ in terms of $\int f(x)$.

7. (a) Find $\int e^x \sin x$.

   (b) Part (a) seems to prove that the only antiderivative of $e^x \sin x$ is $F(x) = \frac{e^x(\sin x - \cos x)}{2}$, however we know that $F(x) = \frac{e^x(\sin x - \cos x)}{2} + C$ is also an antiderivative for any number C. Where does C come from?

ESP Math 408C   WS#35   Fall 1993

1. Work out the following integrals.

$$a)\ \int \frac{7}{(x-2)(x+5)} \qquad b)\ \int \frac{e^{2x}-e^{x}}{e^{x}+1}$$

$$c)\ \int \frac{x^5-x^4-3x^3-2x^2+2x+5}{x^4+x^3} \qquad d)\ \int \frac{x^2-7x+9}{x^3-3x+2}$$

$$e)\ \int \frac{dx}{x^4-16} \qquad f)\ \int \frac{x^4+7x^3-5x^2+13x-10}{(x^2+2x+3)(x-1)^2}$$

$$g)\ \int \frac{5x^2+7x+2}{x^3+2x^2-2x+3} \qquad h)\ \int \frac{dx}{(x^2+16)^2}$$

$$i)\ \int \frac{2x^4+x^3+7x^2+2}{x^5+2x^3+x}$$

2. Evaluate the following integrals, where a, b, and c are constants, $a > 0$.

$$a)\ \int a^{bx} \qquad b)\ \int \frac{dx}{bx+c} \qquad c)\ \int (bx+c)^n$$

$$d)\ \int \frac{\ln(ax)}{bx} \qquad e)\ \int \frac{dx}{a^2+(bx)^2} \qquad f)\ \int \tan(bx+c)$$

3. Prove that the area of the portion of the sphere shown below is $2\pi rh$. *(Notice that this depends only on h, not on the position of the planes!)*

4. It is known that m parts of chemical A combine with n parts of chemical B to produce a compound C. Suppose that the rate at which C is produced varies directly with the product amounts of A and B present at that instant. Find the amount of C produced in t minutes from an initial mixing of A_0 pounds of A with B_0 pounds of B, given that

   (a) $n = m, \ A_0 = B_0,$ and A_0 pounds of C are produced in the first minute.

   (b) $n = m, \ A_0 = \frac{1}{2}B_0,$ and A_0 pounds of C are produced in the first minute.

   (c) $n \neq m, \ A_0 = B_0,$ and A_0 pounds of C are produced in the first minute.

5. Sketch the graph of $y = \frac{1}{x^2-9}$ from $x = 0$ to $x = 3$

   (a) Find the volume of revolution when the area bounded by the graph, the x-axis, and the lines $x = 0$ and $x = 2$ is rotated about the x-axis.

   (b) Find the volume of revolution when the bounded area from (a) is rotated about the line $y = -2$.