

Math 475 Big Problems, Batch 2

Big Problem 7: Tulie Number Flipping.

Tulie has just discovered the Reciprocal button on her calculator (it's labeled " x^{-1} " on the TI-83+). First she noticed that if you enter a number and take the reciprocal, and then take the reciprocal of the result, you get back the original number.

This was a little boring, so she tried the following. First she picked some positive number. Then:

- (a) She subtracted the whole number part from it. If the result was zero, she stopped.
- (b) If it wasn't zero, she took the reciprocal. The result always had a whole number part so she went back to step (a).

For instance: She started with the number 1.4 (she typed in 1.4 and hit Enter).

- a. She subtracted 1 to get .4
- b. Then she took the Reciprocal to get 2.5.
- a & b again. Now she subtracted 2 to get .5, and hit the Reciprocal button to get 2.
- a again. Now she took away 2 and got 0, so she stopped.

She discovered that a lot of numbers she tried were dull because she would eventually get zero in step 'a again' and have to stop. But some numbers like $\sqrt{3}$ were endless, because she never seemed to get zero.

1. Prove $\sqrt{3}$ is really endless.
2. Which choices of natural numbers n make \sqrt{n} an endless number?
3. Tulie discovered that for some numbers n the expansion of n started repeating after only two steps. Which numbers are these? Do you see any patterns in their expansions and in the expansions of nearby numbers? Test some of your patterns.
4. What happens to e ? Can you find a pattern to what happens to $e^{1/n}$?

Big Problem 8: Irrationality of e .

Let us use a famous series expression for e to prove e is irrational.

1. Write down the Taylor Series for the function e^x . You can look it up... I'll assume you derived it in Calculus 2 and can prove it converges everywhere.
2. Write down an infinite series expression for e .
3. Assuming e is genuinely equal to the infinite series, prove e is irrational in the following way: if $e = p/q$, multiply through by $(q!)$. This should lead to an integer part plus something you can prove is less than 1.)

Big Problem 9: Powerful Difference Tables. Recall the *Newton forward difference equation*. It's more powerful than you'd think.

1. Fix n . Write $C(n, i)$ as a polynomial with variable n of degree i for $i = 0, 1, 2$ and 3.

2. The n th tetrahedral number is the sum of the first n triangular numbers. (It's also the number of balls in the first n layers of a tetrahedral ball pyramid.) Find a closed (non-recursive) formula for it using the forward difference equation and (1).
3. Write a formula for the sum of the first n squares using the same difference table analysis.
4. Write a formula for the sum of the first n cubes using the same difference table analysis.

Big Problem 10: Closed Form Formula for the Fibonacci Sequence.

1. We call a sequence A_n *Fibonacci-esque* if it has the property $A_{n+2} = A_{n+1} + A_n$.
2. Find all geometric sequences that are Fibonacci-esque. That is, all numbers C such that the sequence $A_n = kC^n$ is Fibonacci-esque (for constant k).
3. Prove linear combinations of Fibonacci-esque sequences have to be Fibonacci-esque.
4. Find a closed-form formula for the famous Fibonacci sequence $1, 1, 2, 3, 5, 8, \dots$ by finding a sum of Fibonacci-esque geometric sequences that has the famous values.