Math 475 Homework 6
DUE MARCH 19TH, 2010

Order of Operations. What is "the order of operations"? It is a convention to interpret ambiguous arithmetic statements by assigning operations an order of evaluation.

In fact, one can always use matching parentheses to group arithmetic operations into unambiguous pairs. However, in practice this quickly gets annoying and it is hard to match large numbers of parentheses.

The standard convention for order of operations is to group expressions in the following order:

- **Parentheses**. Group every expression between a left and right parenthesis which contains no parentheses.
- **Exponents**. Group every pair of numbers directly connected by exponentiation. In strings where there are more than two such numbers in a row, evaluate the exponentiation from **right to left**. E.g. $2^{3^{5^7}} = 2^{(3^{(5^7)})}$.
- **Negative Sign**. Group a negative sign with the adjacent expression. E.g. $-x+1 = ({}^-x)+1$. Notably, ${}^-2^2 = {}^-(2^2) = {}^-4$, since the Exponents law came first. More importantly, $-x^2 = -(x^2)$.
- **Multiplication and Division**. Group any pair of numbers directly connected by $\times$ or $\div$. For shorthand, we will call this "evaluating the $\times$ and $\div$" and we will use similar terminology in the future. If more than two numbers in a row are connected by $\times$ or $\div$, evaluate them from left to right. E.g. $2 \div 3 \times 4 = (2 \div 3) \times 4$.
- **Addition and Subtraction**. Evaluate all the $+$ and $-$ from left to right, so $10 - 3 + 7 = 14$.

1. Using the *PEMDAS* convention above, re-write these expressions unambiguously with parentheses and evaluate them numerically.
   (a) $1 - 2 + 3 \div 4 \times 5 + 6 - 7$
   (b) $1 + 2 - 3 \times 10 \div 2 \times 9 \div 3$

2. There is no *mathematical* reason for the *PEMDAS* convention. (There are reasons of culture and tidiness of polynomials.) For instance, we could have declared the rule to be to evaluate all operations from left to right, or we could have given a different order of operations. In the problems below, I have given you some clues to figure out my secret new rules for evaluating ambiguous expressions. In each case, rewrite the expression with parentheses to get the target number and state clearly what the rules are. Make sure your rules can unambiguously evaluate any expression with numbers and $+, -, \times, \div$.
   (a) $1 - 2 + 3 \div 4 \times 5 + 6 - 7$ is $1\frac{1}{2}$ and $2 + 2 - 3 \times 10 \div 2 \times 9 \div 3$ is 15.
   (b) $1 - 2 + 3 \div 4 \times 5 + 6 - 7$ is $-1\frac{3}{16}$ and $2 + 2 - 3 \times 10 \div 2 \times 9 \div 3$ is -1.
   (c) $1 - 2 + 3 \div 4 \times 5 + 6 - 7$ is $-14\frac{3}{20}$ and $2 + 2 - 3 \times 10 \div 2 \times 9 \div 3$ is $3\frac{24}{54}$.
   (d) $1 - 2 + 3 \div 4 \times 5 + 6 - 7$ is 2 and $2 + 2 - 3 \times 10 \div 2 \times 9 \div 3$ is 15.
   (e) $1 - 2 + 3 \div 4 \times 5 + 6 - 7$ is $-\frac{4}{16}$ and $2 + 2 - 3 \times 10 \div 2 \times 9 \div 3$ is $\frac{10}{54}$.

3. **Royal Forks and Knives Problem**. The Queens pantry has one drawer that contains forks and knives. The Royal servants take some of them out to set the table for a Royal dinner. Every individual table setting has exactly one fork and one knife. The servants use 2/3 of the forks and 3/5 of the knives in the drawer.

   (a) What is the fraction of the total number of knives and forks that are being used for the Royal dinner?

   (b) Find a general formula expressing the fraction of the total number of knives and forks in use in terms of $\frac{a}{b}$, the fraction of forks in use, and $\frac{c}{d}$, the fraction of knives in use.

4. Constructively prove that between any two distinct fractions there is another fraction. That is, I want a formula for such a fraction given the surrounding fractions $\frac{a}{b}$ and $\frac{c}{d}$.

5. **Multiplying Mixed Fractions**. Draw a picture of $4\frac{1}{2} \times 2\frac{1}{3}$ using an area model by writing each mixed fraction as a sum of a whole number part and a proper fraction part. Find a tidy algorithm for multiplying improper fractions, and write it in the general case of $A\frac{b}{c} \times D\frac{e}{f}$. Use it to calculate $4\frac{1}{2} \times 2\frac{1}{3}$.

6. Use the part-whole model of fractions to justify the subtraction formula for $\frac{a}{b} - \frac{c}{b}$, where $a, b, c \in \mathbb{N}$ and $a > c$ and $b \neq 0$. Use this result to justify the general subtraction formula for $\frac{a}{b} - \frac{c}{d}$.