Math 475 Big Problems, Batch 2

Big Problem 6: Closed Form Formula for the Fibonacci Sequence.

1. We call a sequence A_n *Fibonacci-esque* if it has the property $A_{n+2} = A_{n+1} + A_n$.
2. Find all geometric sequences that are Fibonacci-esque. That is, all numbers C such that the sequence $A_n = kC^n$ is Fibonacci-esque (for constant k).
3. Prove linear combinations of Fibonacci-esque sequences have to be Fibonacci-esque.
4. Find a closed-form formula for the famous Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$ by finding a sum of Fibonacci-esque geometric sequences that has the famous values.

Big Problem 7: Irrational Numbers.

1. Use an argument similar to Euclid's method to prove $\sqrt[n]{2}$ is irrational for any natural number $n > 1$.
2. Use an argument similar to Euclid's method to prove that $\sqrt{p}$ is irrational for p prime.

Big Problem 8: Power Set Sizes.

1. A set S has N distinct elements. How many distinct subsets are there? (Be sure to include S and $\{\}$.) Explain why your formula works.
2. The power set of S, written $P(S)$, is the set of subsets of S. Prove that the cardinality of $P(S)$ is strictly larger than the cardinality of S when S is finite.
3. Consider the general case of a possibly infinite set S. To deal with comparing cardinalities of infinite sets, we say that sets A and B "have the same cardinality" if and only if A fits into B and B fits into A. Here we define "A fits into B" to mean there is an injection from A to B.
4. Consider the power set of the rational numbers, $P(\mathbb{Q})$. Show that the real numbers fit into $P(\mathbb{Q})$. Use this fact, and our previously established uncountability of the reals, to give a proof that $P(\mathbb{Q})$ does not fit into $\mathbb{Q}$.

   Note: There is a theorem called the Bernstein-Schroeder Theorem which says that A and B have the same cardinality if and only if there is a bijection between them. You are allowed to use that theorem for this problem set.
5. Prove that the cardinality of $P(S)$ is always larger than the cardinality of S, even when S is infinite! (*Hint*: Suppose there is a bijection $f : S \to P(S)$. Now consider $\{s \in S \mid s \notin f(s)\}$.)