Big Problem 1. If you have as many 3¢ stamps and 5¢ stamps as you want, you could make 9¢ postage, and 11¢ postage. (How?) But you can't make 1¢ or 2¢ in postage. Let's call postage like 9¢ or 11¢ possible postage for 3¢ and 5¢ stamps, and let's call 1¢ and 2¢ impossible postage.

1. Find the highest impossible postage for 3¢ and 5¢ stamps.

2. Write a convincing argument that it is the highest impossible postage.

3. Now experiment with different pairs of N¢ and M¢ stamps. Find a method, for any specific pair of N¢ and M¢ stamps, for determining whether there is a highest impossible postage, and if there is a highest one, a formula for what it is.

4. Prove your formula is valid for any pair of positive postages.

Big Problem 2.

1. Give me an algorithm for adding two numbers in base $2i$. What is the analogue to our carrying in base 10?

2. Give me an algorithm for multiplying two numbers in base $2i$.

3. Explain why your algorithms work.

Big Problem 3: The Taxman

The Taxman is a one player game invented by Diane Resek. You, the player, will be given a number such as 17, and you should write all the numbers from one through your number on the paper (1 through 17 in this case). Then make a table with one column for the Taxman and one for the player.

First the player picks a number, writes it in her column, and crosses it off the list. Next, she takes all the divisors of that number that have not been crossed off, and writes them in the Taxman's column. Then she crosses those numbers off the list.

Now, she picks another number that has not been crossed off, writes it in her column, crosses it off the list, gives the divisors that haven't been crossed off yet to the Taxman, and crosses the divisors off the list.

The game continues in this way. There is just one rule to follow:

The Rule: Every number the player picks must have at least one divisor that has not been crossed off the list yet. That is, the Taxman must get something on every turn!

When the player can pick no more numbers while following the rule, the Taxman gets all the numbers that are left. The numbers in each column are then added to give the Player's and Taxman's total scores. The one with the highest score wins.

1. Play Taxman up to 10 (that is, use the numbers 1 through 10) and beat the Taxman.

2. Find out what is the maximum amount you can win in a game up to 10. Explain clearly how you know that the amount you write is really the maximum.

3. Play Taxman up to 25 and beat the Taxman. Show your winning game.

4. Find the maximum amount you can win given 25 and provide a convincing argument for why your answer is correct.

Big Problem 4: Egyptian Fractions. Egyptians really liked fractions of the form $1/n$ (unit fractions). What did they do when writing $3/5$? $1/5 + 1/5 + 1/5$ seems ugly and repetitive. We want to use *unit fractions with no repeated denominators*. So, we can write it like $1/2 + 1/10$.

1. Show every **proper** fraction has an Egyptian fraction representation. Be sure to show an explicit algorithm and prove that it terminates with no repeated denominators.

2. Which unit fractions can be written as a sum of TWO unit fractions?

Big Problem 5: Cube of Many Colors.
Twenty-seven small cubes (identical in size) can be used to form a large cube. Is it possible to paint each face of each small cube with one of the three colors (red, green, and yellow) in such a way that (for the same paint scheme):

- in one way of assembling the large cube, all six faces are red,

- in another way of assembling the large cube, all six faces are green,

- in still another way of assembling the large cube, all six faces are yellow?

You get part credit for proving it's possible. You get the rest of the credit for giving me an explicit paint scheme and rearrangement scheme.

Big Problem 6: Repeating Decimal Periods. When a decimal representation repeats infinitely, we call the length of the repeating block "the period".

1. Examine the periods of fractions of the form $1/p$, where p is a prime. Discover a cool rule describing the periods in relation to p, and share it with me. Explain why it is so.

2. Generalize your rule for reciprocals of prime numbers to the case of a general fraction, $1/n$, where n is a natural number.