1. (**do, but don't turn in**) Consider this model of the integers. Every integer is an IOU, either a credit or debt in dollars. So 20 means we are owed \$20 (a \$20 credit), and -20 means we owe \$20 (a \$20 debt). Addition means putting together and subtracting something means to remove it. Explain in words using this model what is a reasonable value for:

   (a) 20 + (-20)

   (b) 20 - (-20)

   (c) (-20) + (-20)

   (d) -20 + 0

2. (**do, but don't turn in except for (2i)**). Modeling multiplication is more complex. Let's model $M \times N$ as follows. For a positive N, we say it's getting N IOUs each worth $\$M$. For a negative N, we say it's losing N IOUs each worth $\$M$. Explain in words using this model what is a reasonable value for:

   (a) 20×1

   (b) 1×20

   (c) 20×0

   (d) 0×20

   (e) 20×-3

   (f) $(20 \times 3) + (20 \times (-3))$

   (g) -20×3

   (h) -20×-1

   (i) **(turn in)** -20×-3

3. **Defining the Integers from the Natural Numbers: Addition.** Suppose we have constructed the natural numbers to our satisfaction along with a binary addition operation. Note there is no "subtraction" operation or "opposite" operation yet, so this is an approach independent from the "opposite adjoining" in the last homework.

   Here we will use what is called the *Grothendieck Construction*. This is a very general construction which actually applies to any structure with a commutative, associative binary operation and an identity element. It takes such a structure and embeds it in a larger structure that extends the operation so that every element now has an inverse (i.e. embeds it in an abelian group). The set of natural numbers fits these preconditions, and we'll discuss this specific case, but this exact argument works for many other sets.

One defines new numbers of the form "$a-b$" where a and b are natural numbers. It is tempting to just see "3-2" as "1" because of habit, but we're not assuming we know anything about subtraction yet. So for now we just regard "$a-b$" as an ordered pair written with panache.

And yet $3-2$ really ought to be the same as $7-6$. Thus, we declare "$a-b$" to be equivalent to "$c-d$" if and only if $a+d=b+c$ (notice how we only use addition in the definition). We write this $a-b \sim c-d$

In fact, this is a genuine equivalence relation, so each ordered pair is really an equivalence class of expressions "$a-b$". These ordered pairs are called the integers and the natural numbers are embedded in the integers as the classes "$a-0$".

(a) (Do, don't turn in) Prove to yourself that the relation defined above is really an equivalence relation.

(b) What is the natural way to define an addition on these objects? I.e. $(a-b)+(c-d)=?$, where your answer is another difference pair.

(c) Verify that your addition is well-defined. That is, if $(a-b) \sim (a'-b')$ and $(c-d) \sim (c'-d')$ then $(a-b)+(c-d) \sim (a'-b')+(c'-d')$.

(d) What is the additive identity? Prove it.

(e) What is the opposite of $(a-b)$? Prove it.

(f) Give a clean description of the set of negative numbers (the opposites of the natural numbers)?

4. **Defining the Integers from the Natural Numbers: Multiplication**. Once the integers are constructed with an addition, it's natural to want to extend multiplication.

(a) What is the natural way to define multiplication on pairs of integers? I.e. $(a-b)(c-d)=?$.

(b) Check that your multiplication is well-defined, that is if $(a-b) \sim (a'-b')$ then $(a-b)(c-d) \sim (a'-b')(c-d)$. Technically you should also check that $(a-b)(c-d) \sim (a-b)(c'-d')$ but I don't need to see that (similar) calculation.

(c) Prove that this new integer multiplication is commutative assuming natural number multiplication is commutative.

(d) Check that 0 times anything is 0.

(e) Check that two negative numbers multiply to a positive product.

5. (don't turn in) Divide 128 by 7 using relaxed repeated subtraction.

6. Explain why in class we called our long division algorithm abbreviated uptight repeated subtraction. Explain this using the example of 128 divided by 7.

7. Describe 128 divided by 7 using

   (a) a partitive model (sharing)

   (b) a measurement model (taking out bundles)

8. Henry spends 3/14 of his monthly income for rent and 2/11 of what is left on food. If he has \$540 left, what is his monthly income ? (Hint: Pictures really help).