Student Errors: What Can They Tell us About What Students DO Understand?

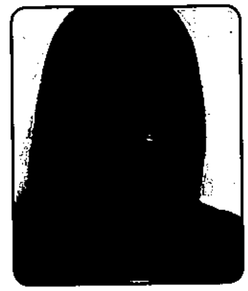
By Julie McNamara and Meghan M. Shaughnessy

Students' challenges with fractions are well documented. On the 2007 National Assessment of Educational Progress (NAEP) test, fewer than half of grade 8 students (49%) were able to correctly identify which arrangement of the fractions 2/7, 1/2, and 5/9, showed the fractions ordered from least to greatest. On the 2009 NAEP test, only 25% of grade 4 students provided the correct answer (5/8) to the following question: Which fraction has a value closest to 1/2? (Answer choices were 5/8, 1/6, 2/2, and 1/5.) These findings are not surprising to many teachers, particularly those who teach middle school mathematics. Research indicates understanding fractions is a "foundational skill essential to success with algebra," (U.S. Department of Education, 2008) and there is a strong positive correlation between students' understanding of fractions and their overall success in mathematics (Gomez, 2009).

It is often tempting to dismiss student errors as merely careless mistakes or a sign of global misunderstanding of a topic. Interpreting errors as careless mistakes often leads teachers to assume the student would produce the correct answer, if given the question on another day. Interpreting the student error as a sign of global misunderstanding of the topic often leads to completely reteaching the content. In many cases, however, student errors and incorrect responses are the result of students' "partial understandings" (Saxe et al, 2010) or correct answers to slightly different questions (Wells & Coffey, 2005).

Instead of considering incorrect responses as errors or mistakes to be avoided, Saxe et al take the position that they are often a normal part of the development of students' understanding of a topic. Exposing and discussing students' partial understandings can be a very productive instructional strategy for deepening and refining students' thinking. Wells and Coffey (2005) discuss how students' incorrect responses may actually be correct answers to related but different questions. For example, both 1/5 and 2/5 could be considered correct answers to the question of how to fairly share two candy bars among five children, depending on what is considered the whole. If the whole is one candy bar, then each child would get 2/5 of a candy bar. On the other hand, if the whole is the candy, then each child would get 1/5 of the candy. A third possibility is that each child would get 1/5 of each candy bar.

Julie McNamara



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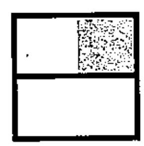
Below are examples of common student errors identified in *Beyond Pizzas and Pies: 10 Essential Strategies for Supporting Fraction Sense* (McNamara & Shaughnessy, 2010). We discuss how these errors are often rooted in student understanding. By approaching student errors from this perspective, teachers can acknowledge what students do know and understand and build on this to guide students to deeper and more robust understanding of complex mathematical topics.

EXAMPLE 1: Write a fraction for the shaded region:



STUDENT RESPONSE: *1/3*

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Student Errors

Continued from previous page

While the student's response of "1/3" would be considered incorrect, it does indicate understanding of conventional fraction notation. By identifying the shaded region as 1/3, the student understands when writing a fraction to name a shaded part of a region, the numerator indicates the number of parts that are shaded (in this case "1") and the denominator indicates the total number of parts that constitute the whole (in this case "3"). This response could be considered an example of the student's partial understanding, in that he/she correctly identifies a relationship between the shaded part and the whole region but does not understand that the relationship between the shaded and unshaded parts is not strictly discrete (i.e., one shaded part out of three parts) but is also proportional (three of the shaded parts would not completely cover the entire region). In addition, the student's response of "1/3" would be correct if the region were divided into three *equal* parts. By acknowledging what the student who answers "1/3" *does* understand about fraction notation, and engaging students in a discussion about whether the shaded region shows 1/3 or 1/4 of the larger square, teachers can help students solidify their understanding of what fractions actually mean.

EXAMPLE 2: **Circle the larger fraction:**

5/6 7/8

STUDENT RESPONSE: *Student circles 5/6 and writes, "If the denominator is smaller, the piece is bigger."*

This response indicates the student understands when a quantity or region is divided into fewer pieces (i.e., six pieces versus eight pieces), each resulting piece will be bigger. This is an idea that is often hard for students to understand, as students initially believe that sixths are smaller than eighths because six is less than eight. In this case, however, the student is only attending to denominators of the fractions, not seeming to realize that in order to compare two fractions, both the numerator and denominator need to be considered. If the question had been changed slightly to read, "Circle the larger fraction: 1/6 or 1/8," or, "Which fraction has the larger denominator, 5/6 or 7/8?" then the student's thinking would net the correct answer. Building on the student's understanding of the relationship between the value of the denominator and the size of the resulting parts, while helping him/her recognize the role of the numerator in understanding fractional values, teachers can help students develop more complete understanding of fraction values.

EXAMPLE 3: **Patrick orders a cheese pizza at Pizza Delight. He eats 1/2 of his pizza. Kevin orders a cheese pizza from Vinny's Pizza. He eats 1/3 of his pizza. Who ate more pizza?**

STUDENT RESPONSE: *Student chooses Patrick and writes, "1/2 is bigger than 1/3."*

This task was designed to assess students' understanding of the role of context in fraction comparison tasks. As with the response shown in Example 2, above, the student who chooses Patrick is applying an important idea about fraction comparison. This student understands that 1/2 of something is greater than 1/3 of something. What this student may not understand is the statement, "1/2 is greater than 1/3" implies the wholes are the same (i.e., 1/2 of a large pizza from Pizza Delight is larger than 1/3 of that same pizza), or the same size (i.e., 1/2 of a large pizza from Vinny's Pizza is larger than 1/3 of another large pizza from Vinny's Pizza). In the example given, this cannot be assumed as the boys ordered pizza from two different shops and the sizes of the pizzas are not provided, thus there is no way to compare the two fractions. Had the question read, "Patrick and Kevin both ordered large pizzas from Tony's Pizza Shop. Patrick ate 1/2 of his pizza. Kevin ate 1/3 of his pizza. Who ate more pizza?" then the response "Patrick" would be correct. By contrasting the question involving two different pizza shops with the one involving both boys ordering the same size pizza from the same shop, teachers can help students identify the information needed to make an accurate comparison.

EXAMPLE 4: **A student does the following multiplication problem:**

$$\frac{5}{6} \times \frac{2}{2} = \frac{10}{12}$$

Look at the statement below:

$\frac{10}{12}$ is twice as large as $\frac{5}{6}$

Decide whether you agree or disagree with the statement.

Agree **Disagree**

STUDENT RESPONSE: *Student agrees with the statement.*

By agreeing with the statement "10/12 is twice as large as 5/6," the student correctly identifies the relationship between the numerators (10 is twice as large as 5) and denominators (12 is twice as large as 6) of the two fractions. The student may also be familiar with the procedure of multiplying by n/n to create an equivalent fraction, but likely does not understand that multiplying by 2/2, while it

... continued on page 31

Student Errors

Continued from page 29

changes the value of each part of the fraction (doubling the numerator and denominator), does not change the value of the resulting fraction. This student may not fully understand that 2/2 = 1, and that multiplying by 2/2 is the same as multiplying by 1, thus creating a new fraction that looks different but has the same value. If the statement were changed to read, "The numerator and denominator in 10/12 are twice as large as the numerator and denominator in 5/6," then the student's answer would be correct. Having students analyze statements like "10/12 is twice as large as 5/6," and modifying them by changing the wording so that they are correct, can help students to refine their thinking and attend to precise use of mathematical language.

Building on the knowledge that students draw upon as they make sense of mathematics is an essential aspect of supporting their learning. By viewing mistakes and student errors as indications of partial understanding or as correct answers to slightly different questions, teachers can use student thinking as a resource to help students deepen and refine their thinking. Incorrect responses can be wonderful starting places for discussion and analysis of important and challenging mathematical ideas. In our work and in the work of our colleagues, we have found engaging in discussions of answers produced by "partial understandings," or by correct answers to somewhat different questions, can be beneficial for all students.

Julie McNamara is currently an education specialist with Math Solutions. She designs and provides professional development, site-based coaching, and ecoaching to K–8 classroom teachers, mathematics coaches, and administrators across the country. She has over 20 years experience as a classroom teacher, a professional development provider, and an elementary mathematics methods instructor throughout the San Francisco Bay Area. She received a PhD in mathematics education from UC Berkeley, focusing her research on the development of students' rational number concepts. She is the co-author of Beyond Pizzas & Pies: 10 Essential Strategies for Supporting Fraction Sense. *She lives in Berkeley, CA with her husband Rick, a fifth-grade teacher.*

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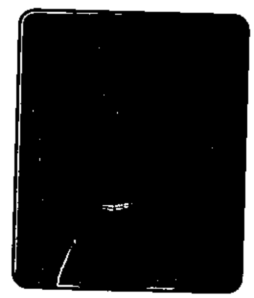
Attention: NCSM Members Planning to Attend NCTM in Indianapolis

By Jerry Cummins, NCTM Representative from NCSM



Jerry Cummins

Show your "LOVE" for NCSM by volunteering two hours of your time at the NCTM Annual Conference in Indianapolis. I know it would be difficult for you to select a two-hour time period to spend in the NCSM booth. So, I am just asking you to commit to be a volunteer for two hours. As we get within three weeks of the NCTM Conference, I will give you an opportunity to select the slot that will fit your schedule.

This is a rewarding opportunity to discuss the benefits of mathematics leaders joining NCSM. Please send me an email ASAP if you are willing to give us two hours of your time.

Jerry Cummins lives in Hinsdale, Illinois. In addition to being our NCTM representative, he is an NCSM Past President, a mathematics consultant, and an author. He can be reached at (630) 323-0959 or by email at jcltmath1@aol.com.

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