trating Discussions

Five practices constitute a model for effectively using student responses in whole-class discussions that can potentially make teaching with high-level tasks more manageable for teachers.

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Discussions that focus on cognitively challenging mathematical tasks, namely, those that promote thinking, reasoning, and problem solving, are a primary mechanism for promoting conceptual understanding of mathematics (Hatano and Inagaki 1991; Michaels, O'Connor, and Resnick forthcoming). Such discussions give students opportunities to share ideas and clarify understandings, develop convincing arguments regarding why and how things work, develop a language for expressing mathematical ideas, and learn to see things from other perspectives (NCTM 2000).

Although discussions about high-level tasks provide important

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opportunities for students to learn what mathematics is and how one does it, they also present challenges to the teacher who must determine how to orchestrate a discussion built from a diverse set of responses. A key challenge is to build on *and* honor student thinking while ensuring that mathematical ideas at the heart of the lesson remain prominent. Take, for example, the Bag of Marbles task in **figure 1**. Although many approaches can be taken to solve it, none are implied or specified in the statement of the task. Students can solve this task in several different ways, as shown in column 1 of **figure 3**, by using their knowledge of fractions, ratios, and percents.

In orchestrating the discussion of the Bag of Marbles task, the teacher must "decide what aspects of a task to highlight, how to organize and orchestrate the work of the students, what questions to ask to challenge those with varied levels of expertise, and how to support students without taking over the process of thinking for them and thus eliminating the challenge" (NCTM 2000, p. 19). Giving students too much or too little support, or too much direction, can result in a decline in the cognitive demands of the task (Henningsen and Stein 1997).

Discussions of solutions to high-level tasks such as the Bag of Marbles, shown in **figure 2**, can also easily become little more than elaborate show-and-tells (Ball 2001; Wood and Turner-Vorbeck 2001), with limited teacher or student commentary. Opportunities can be lost with respect to drawing connections among the methods or tying them to key disciplinary ideas. In this article, we propose a

model for effectively using student responses in whole-class discussions that can potentially make teaching with high-level tasks more manageable for teachers. We hope to help increase the likelihood that the demands of high-level tasks will be maintained during instruction and that the key mathematical ideas to be learned will be emphasized. The five practices that comprise the model will be the focus of the remainder of this article.

THE FIVE PRACTICES MODEL
The five practices are—

1. anticipating student responses to challenging mathematical tasks;
2. monitoring students' work on and engagement with the tasks;
3. selecting particular students to present their mathematical work;
4. sequencing the student responses that will be displayed in a specific order; and
5. connecting different students' responses and connecting the responses to key mathematical ideas.

These practices can help teachers use students' responses to advance the mathematical understanding of the class as a whole. These practices give teachers control over what is likely to happen in a discussion as well as more time to make instructional decisions. This is possible because much of the decision making has been shifted to the planning phase of the lesson. Each practice is described in more detail in the article, using the Bag of Marbles task shown in **figure 1** and related student work in **figure 2**, to ground the discussion.

Anticipating
The first practice is for teachers to anticipate the different ways the

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Fig. 1 The Bag of Marbles task

Ms. Rhee's math class was studying statistics. She brought in three bags containing red and blue marbles. The three bags were labeled as shown below:
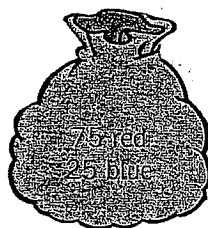
Bag *x*
Total = 100 marbles

Bag *y*
Total = 60 marbles

Bag *z*
Total = 125 marbles

Ms. Rhee shook each bag. She asked the class, "If you close your eyes, reach into a bag, and remove 1 marble, which bag would give you the best chance of picking a blue marble?"

Which bag would you choose?

Explain why this bag gives you the best chance of picking a blue marble. You may use the diagram above in your explanation.

Fig. 2 Student solutions to the Bag of Marbles task

Bag *x* is 1/3 blue. Bag *y* is 1/2 blue. Bag *z* is 1/4 blue. 1/2 is a lot so it must be bag *y*. A	Because bag *y* is 1/3 full of blue marbles and bag *x* is only 1/4 full of blue marbles and bag *z* is only 1/5 full of blue marbles D	Notice in the first bag there are 75 red and 25 blue, that is a 1:3 chance. Notice that in the second bag there are 40 red 20 blue, that is a 1:2 chance. Notice in the third bag there are 100 red 25 blue, that is a 1:4 of a chance. This shows that in bag *y* you would be likely to pick a blue marble. G
I found the % of blue marbles in each bag. *x:* 25/100 = 25% *y:* 20/60 = 33 1/3% *z:* 25/125 = 20% B	Bag *x* is 1/4 blue and bag *y* is 1/3 blue. Better chance bag *y*. Bag *y* has 1 blue to 2 reds, and bag *z* has 1 blue to 4 reds. Better chance bag *y*. ·E	
x: 75/25 = 3/1 = 3 *y:* 40/20 = 2/1 = 2 *z:* 100/25 = 4/1 = 4 Since the marbles in bag *z* total 125, I think your chances would be higher than the others. C	The *x* bag has 75 red and 25 blue. There are 50 extra marbles that are red. The *z* bag has 100 red and 25 blue. There are 75 extra red than blue. Now Bag *x* has 40 red and 20 blue. There are 20 extra red than blue. F	Bag *x* has 75 red and 25 blues and bag *z* has 100 red and 25 blues. In bags *x* and *z* the blues are the same, so then you would have to look at the red to see which is the least between them, and bag *x* has 75 red and 75 is less than 100, so I chose bag *x*. H

mathematical task can be solved. This requires considering how students might mathematically interpret a problem, the array of strategies (both correct and incorrect) they might use to solve it, and how those strategies and interpretations might relate to the mathematical ideas the teacher would like his or her students to learn.

Anticipating solutions requires that the teacher do the problem as many ways as he or she can. However, it is often helpful to expand on what one might be able to do individually by working on the task with other teachers and reviewing responses to the task that might be available (e.g., work produced by students the previous year; responses that are published, along with tasks in supplementary materials). The teacher may also consider consult-

ing research on student learning of the mathematical ideas embedded in the task. For example, research suggests that students often use additive strategies, such as response F in **figure 2**, to solve tasks similar to the Bag of Marbles in which there is a multiplicative relationship between quantities (Hart 1981; Heller et al. 1989; Kaput and West 1994). Anticipating this approach before the lesson begins would make it possible for the teacher to recognize when his or her students produce it and carefully consider beforehand what actions should be taken. In this task, for example, teachers could determine in advance what questions to ask so that students become aware of the multiplicative nature of the relationship between the red and blue marbles. This would help bring up the solution during

discussion so that all students can consider why it appears to work but is not a valid method.

Monitoring

Monitoring student responses involves paying close attention to students' mathematical thinking and solution strategies as they work. One way is by circulating around the classroom while students work either individually or in small groups. According to Lampert (2001, p. 140), paying close attention to what students do as they work makes it possible "to use my observations to decide what and who to make focal" during the discussion that follows.

To further the monitoring process, the teacher can create a list of solutions before teaching the lesson to anticipate what students will produce

Fig. 3 Tool for monitoring students' explorations –

Strategy	Who and What	Order
Fraction Determine the fraction of each bag that is blue marbles (x is 1/4; y is 1/3; z is 1/5). Decide which of the three fractions is larger (1/3). Select the bag with the largest fraction of blue marbles (bag y).		
Percent Determine the fraction of each bag that is blue marbles (x is 25/100; y is 20/60; z is 25/125). Change each fraction to a percent (x is 25 percent; y is 33 1/3 percent; z is 20 percent). Select the bag with the largest percent of blue marbles (bag y).		
Ratio (Unit Rate) Determine the part-to-part ratio that compares red with blue marbles for each bag (x is 3:1; y is 2:1; z is 4:1). Determine which bag has the fewest red marbles for every 1 blue marble (bag y)		
Ratio (Scaling Up) Scale up each bag so that the number of blue marbles in each bag is the same (e.g., x is 300 R and 100 B; y is 200 R and 100 B; z is 400 R and 100 B). Select the bag that has the fewest red marbles for 100 blue marbles (bag y).		
Additive Determine the difference between the number of red and blue marbles in each bag (x is 50; y is 20; z is 75). Select the bag that has smallest difference (bag y).		
Other		

so that the mathematical goals for the lesson can be accomplished. The list, such as that in **figure 3**, can help the teacher track which students or groups produced which solutions or what ideas were brought out that he or she wants to capture during the whole-group discussion. The cell labeled "other" in the first column gives the teacher the opportunity to capture ideas that had not been anticipated. (Column 2 in **fig. 4** provides an example of what might be recorded by a teacher during the monitoring phase in a classroom in which students produced the solutions shown in **fig. 2**.)

It is important to note, however, that monitoring involves more than just watching and listening to stu-dents. During this time, the teacher should also ask questions that will make students' thinking visible and help students clarify their thinking. The teacher should also ensure that all members of the group are engaged in the activity and press students to consider aspects of the task to which they need to attend.

Many of these questions can be planned in advance of the lesson, based on the anticipated solutions. For example, consider a teacher who anticipated that a student would use a unit-rate approach (see solution C in **fig. 2**), in which the number of red marbles in each bag is compared with 1 blue marble. The teacher might be prepared to ask the student about the meaning of the numbers 3, 2, and 4 and how these numbers provide some insight into the bag so that someone would have the best chance of picking a blue marble. Questioning a student or group of students while they are exploring the task gives them the opportunity to refine or revise their strategy before launching a whole-group discussion.

Selecting
Having monitored the work of students as they explore the task, the teacher can then select particular stu-dents to share their work with the rest of the class to get "particular pieces of the mathematics on the table" (Lampert 2001, p. 140). Selecting particular students and their solutions is guided by the mathematical goal for

Fig. 4 Sample of how the monitoring tool could be used

Strategy	Who and What	Order
Fraction	Solution D Student does not explain how he or she determined that 1/3 was the largest fraction. Solution A Student created ratios but treats them as though they were fractions.	Second
Percent	Solution B Student does not explain which bag gives you the best chance and why.	Third
Ratio (Unit Rate)	Solution C Student found the unit rates but did not use this information to answer the question. Solution G Student compares blue to red ratio but does not explain why a 1:2 chance is better than the others.	Fourth
Ratio (Scaling Up)	No one used this approach.	
Additive	Solution F Student picked the right bag but for the wrong reason.	First
Other Reasoning	Solution H Student compares x and z and concludes that x gives a better chance but does not consider bag y.	
Other Combination of fraction and ratio	Solution E Student combines approaches seen in response D (compares the fractions 1/4 and 1/3), then switches to compare part-to-part ratios as in solution G.	

the lesson and the teacher's assessment of how each contribution will contribute to that goal. Teachers may want to keep track of which students present their work, so that all students have the opportunity to share their thinking publicly.

A teacher might have two mathematical goals for a lesson based on the Bag of Marbles task:

1. Students need to understand that to compare bags of marbles, they need a common basis for comparison.
2. Students need to be able to distinguish different types of comparisons (i.e., part-to-part, part-to-whole, percents).

To reach these goals, a teacher might want to select solutions that use different types of comparisons, such as fractions (e.g., solution D, which shows a part-to-whole relationship), percents (e.g., solution B, which shows part-to-whole comparison), and ratios (e.g., solution G, which shows a part-to-part relationship).

In addition, a teacher might also want to share a solution that uses an incorrect additive comparison, such as that in solution F. The teacher may also wish to share a solution that was not produced by any student in the class to introduce a strategy that he or she would like students to learn, such as the Scaling Up strategy shown

in **figure 3**. In this case, the teacher might present the strategy as one produced by a student in another class or in another year and invite the class to determine its validity. Although all unique approaches could be shared, the teacher needs to carefully consider the amount of time available for the discussion, the value added by each contribution, and the mathematical story line that will provide coherence to the discussion.

Sequencing
By selecting particular students who will be presenting their work to the class, the teacher can make decisions regarding how to sequence the

presentations. By making purposeful choices about the order in which students' work is shared, teachers can maximize the chances that their mathematical goals for the discussion will be achieved. For example, the teacher might want to have a student present the strategy used by the majority before one that only a few students used. This will validate the work that students did and make the beginning of the discussion accessible to as many as possible.

Alternatively, the teacher might want to begin with a strategy that is more concrete, using drawings or concrete materials, and move to strategies that are more abstract, using algebra. This approach validates less-sophisticated approaches and allows for connections between the concrete and the abstract. If a common misconception underlies a strategy that several students used, the teacher might want to address it first so that the class can clear up that misunderstanding and develop more successful ways of tackling the problem.

Finally, the teacher might want to have contrasting or related strategies presented right after one another to make it easier for the class to compare them. Again, sequencing anticipated responses is key to the lesson. Unanticipated responses can then be fit into the sequence as the teacher makes final decisions about what is to be presented.

It is important to note that there is not one right way to select and sequence a set of responses. The selection and sequence depends in large measure on the teacher's goals for a lesson. In the Bag of Marbles task, the goal may be for students to understand that to compare bags of marbles, a common basis for comparison is needed. Perhaps students should also be able to distinguish different types of comparisons. The approach would include the decision to sequence the

Questioning a student or group of students while they are exploring the task gives them the opportunity to refine or revise their strategy before launching a whole-group discussion

solutions in this order (see column 3 in **fig. 4**):

1. F (incorrect additive)
2. D (correct but incomplete fraction)
3. B (correct but incomplete percent)
4. G (correct but incomplete ratio)

Beginning with solution F allows the teacher to clear up a common misunderstanding and give the class a chance to consider why this approach does not work. Establishing the need for a common basis for comparison then provides a lens through which to view the remaining solutions. Solution D uses a fraction approach in which each numerator is 1, giving a common basis for comparison. Solution B builds on the fraction work in solution D, but instead of reducing the original fractions to unit fractions, the fractions are converted to percents. The equivalence of all three forms (unit fractions, unreduced fractions, and percents) can be discussed. Solution G is similar to solutions D and B in that there is a common basis for comparison—1 blue marble was compared with the number of red marbles—yet it represents a part-to-part rather than a part-to-whole relationship.

Connecting
Finally, the teacher helps students draw connections between their solutions and other students' solutions as well as the key mathematical ideas in the lesson. The teacher can help students make judgments about the consequences of different approaches for the

range of problems that can be solved. Effective discussions can help students evaluate accuracy and efficiency in solving such problems, and the kinds of mathematical patterns that can be most easily discerned. Rather than having mathematical discussions consist of separate presentations of different ways to solve a particular problem, the goal is to have student presentations build on each other to develop powerful mathematical ideas.

For example, consider the Bag of Marbles scenario in which the teacher has decided to select and sequence solutions F, D, B, and then G. The teacher can begin by asking each student presenter to clarify how he or she knew that bag y would provide the best chance of selecting a blue marble, since the students who produced solutions D, B, and C fell short of explaining how they used mathematics to select bag y. The teacher might then want students to compare each newly presented solution with the other solutions and determine how they are the same *and* different. For example, comparing solution G with solutions D and B would highlight the fact that G is a part-to-part comparison, whereas D and B are part-to-whole comparisons.

Although a teacher cannot know with 100 percent certainty how students will solve a problem before a lesson begins, many solutions can be anticipated. As a result, the selection and sequencing, as well as specific connections, can be planned ahead of time. By planning what is predictable

before the lesson, the teacher is then better prepared to incorporate the solutions that were not anticipated.

CONCLUSION

The premise underlying this article is that identifying and using the five practices discussed earlier can make discussions of cognitively challenging tasks more manageable for teachers. By giving teachers a roadmap that they can follow before and during whole-class discussions, these practices have the potential for helping teachers more effectively orchestrate discussions that are responsive to both students and the discipline. The five practices can also help teachers establish a sense of efficacy over their instruction (Smith 1996); they learn that there are ways for them to reliably shape students' discussions to move them toward more powerful mathematical understandings.

We see these five practices as a focused model for planning a successful whole-class discussion that builds on student thinking while continuing to advance the teacher's mathematical goals for the lesson. As such, it is intended to help teachers keep "an eye on the mathematical horizon" (Ball 1993), never losing sight of what they are trying to accomplish mathematically.

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