answer of 49 mentally. For $2\frac{1}{4} \times 16$, you could eliminate the fraction by multiplying $2\frac{1}{4}$ by 4 and dividing the 16 by 4. This lets us express the problem as 9×4, or 36. Try using this strategy to compute $3\frac{1}{5} \times 45$: eliminate the fraction by multiplying $3\frac{1}{5}$ by 5 and then divide 45 by 5. Now the problem is 16×9. You could keep on halving and doubling with the whole numbers: $16 \times 9 = 8 \times 18 = 4 \times 36 = 2 \times 72 = 144$. Or, since you know that $16 \times 10 = 160$, all you have to do is subtract the extra 16 to get the answer of 144.

You can also use this strategy to eliminate decimal numbers. How about 0.8×350? If you multiply the 0.8 by 10 and divide the 350 by 10, you express the problem as 8×35. Halving and doubling, you get 4×70, which equals 280. Or you could think of 0.8 as $\frac{4}{5}$, expressing the decimal as a fraction; $\frac{4}{5}$ of $350 = 4 \times 70$. All these strategies work because of the associative property of multiplication. You can group the factors in clever ways first to make the computation easier:

$$0.8 \times 350 = ?$$
$$0.8 \times (10 \times 35) = (0.8 \times 10) \times 35 = 8 \times 35$$
$$(4 \times 2) \times 35 = 4 \times (2 \times 35) = 280$$
$$0.8 \times 350 = (4 \times \tfrac{1}{5}) \times 350$$
$$4 \times (\tfrac{1}{5} \times 350) = 4 \times 70 = 280$$

Note how quickly all these alternative, creative ways to compute products can be done—in most cases mentally. If paper and pencil are used, it is only to keep track. The ability to explore numbers like this is the result of a deep understanding of number, landmark numbers, properties, and operations. And it characterizes true numeracy. In contrast, students who are asked to practice the algorithm over and over to multiply 0.8×350 stop thinking. They sacrifice the relationships in order to treat the numbers simply as digits. And any teacher of middle school students will attest to the difficulties students have as they try to complete each of the multiplication steps, regroup appropriately, and determine where the decimal point goes in the answer.

Does this mean that students don't need to know how to calculate? Of course not. To be successful in today's world, they need a deep understanding of mathematics. They will be bombarded with numbers, statistics, advertisements, and similar data every day—on the radio, on television, and in newspapers. They will need good mental ability and good number sense in order to evaluate advertising claims, estimate quantities, efficiently calculate numbers and judge whether these calculations are reasonable, add up restaurant checks and determine equal shares, interpret data and statistics, and so on. They need to be able to move back and forth from fractions to decimals to percents. Students need a much deeper sense of number and operation than ever before—one that includes algorithms, but emphasizes mental arithmetic and a repertoire of strategies that allows them to both estimate and make exact calculations mentally.

How do we, as teachers, develop students' ability to do this? How do we engage them in learning to be young mathematicians at work? The minilessons in this guide will help students to develop a deeper understanding of the computations they perform.

Using Minilessons to Develop Number Sense: An Example

Minilessons are usually done with the whole class together in a meeting area. Young students often sit on a rug; for older students, benches or chairs can be placed in a U-shape. Clustering students together like this, near a chalkboard, is helpful because you will want to encourage pair talk at times, and you will need space to represent the strategies that will become the focus of discussion. The problems are written one at a time and learners are asked to determine an answer. Although the emphasis is on the development of mental arithmetic strategies, this does not mean learners have to solve the problems in their heads—but it is important for them to do the problem with their heads! In other words, encourage students to examine the numbers in the problem and think about clever, efficient ways to solve it. The relationships between the problems in the minilesson will support students in doing this. By developing a repertoire of strategies, an understanding of the big ideas underlying why they work, and a variety of ways to model the relations, students are constructing powerful toolboxes for flexible and efficient computation. Enter a classroom with us and see how this is done.

Each day at the start of math workshop, Dawn Selnes, a fifth-grade teacher in New York City, does a short minilesson on computation strategies. She usually chooses a string of five or six related problems (like the ones provided in this resource unit) and asks the students to solve them, one at a time, sharing their strategies with each other. She allows her students to construct their own strategies by decomposing numbers in ways that make sense to them. Posted around the room are signs the students made throughout the year as they developed a repertoire of strategies for operations with fractions. The signs read, "Make use of money," "Use clocks," "Halve and double," "Eliminate the fractions."

On the chalkboard today is the first problem in Dawn's string:

9 × 30

15 × 18

4 ½ × 60

2 ½ × 120

15 × 36

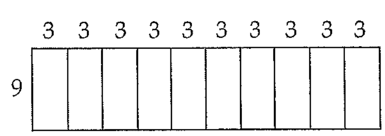
15 ½ × 36

5 ¼ × 40

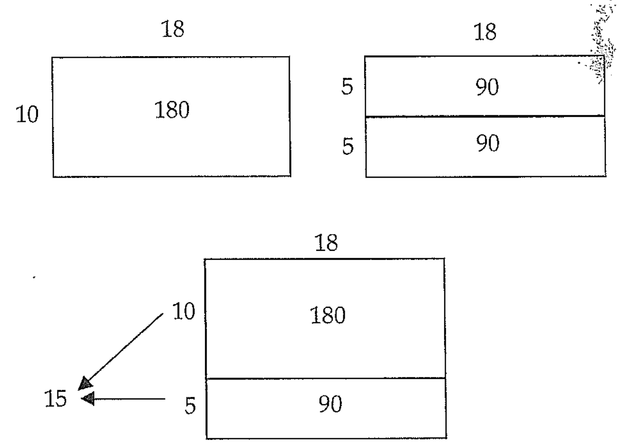
0.8 × 80

Although the string progresses to fractions and decimals, it begins with a few whole number multiplication problems, and one student, Alice, describes how she solved 9 × 30. "I just used all the factors," she explains. "I thought of it as 9 times 3 times 10. I knew that 9 times 3 is 27, and 27 times 10 is 270."

Dawn draws ten 9 × 3 arrays (showing how they fit together to make a 9 × 30) to represent her strategy and then asks for other strategies.

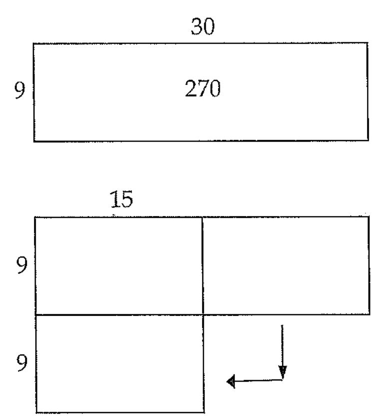


Most of the students have treated the problem similarly, so Dawn goes to the next problem in the string, 15 × 18. Several students use the distributive property here. Tom's strategy is representative of many, and several students nod in agreement as he explains how he multiplied 10 × 18 and got 180 and then took half of that to figure out the answer to 5 × 18. He completes the calculation by adding 180 to 90, for an answer of 270. Dawn draws these arrays:



Lara's strategy is similar, perhaps not as elegant, but it makes sense to her. She multiplies using tens, too, but she breaks up the 18 instead of the 15 and multiplies 10 × 15, and then 8 × 15. These two products together also result in 270, and Dawn represents them with an array model.

Ned agrees with that answer but with a smile he says, "Yeah, but you didn't even have to calculate. It's the same as 9 × 30, because the 30 is halved, and the 9 is doubled!" Dawn draws an array, cuts it in half, and the students discuss how the pieces can be moved to show the equivalence.



Although all the students in the class are comfortable with this doubling and halving strategy and understand why it works, they have not all thought to use it, because Dawn has turned the numbers around. It might have been more obvious if she had written 18 × 15 directly underneath 9 × 30. But she wants to challenge them to think.

Now Dawn moves to fractions. She writes $4\frac{1}{2} \times 60$ as the third problem. Several students immediately raise their hands, but Dawn waits for those still working to finish. Alice is one of them, so she asks her to share first. "I split it into 4×60 first," Alice begins, "and I did that by doing $4 \times 6 = 24$. Then 24 times 10 is 240. Then I knew that a half of 60 is 30. So 30 plus 240 is 270."

"My way is kind of like yours," another classmate, Daniel, responds, "but I subtracted."

"But then you would get the wrong answer," Alice tells him, looking puzzled.

"No, what I mean is I did $5 \times 6 \times 10$. That was 300. Then I subtracted the 30."

"Where did you get the 5?" Several of his classmates are also now puzzled.

"That was easier for me than $4\frac{1}{2}$. But that's why I took 30 away at the end," Daniel explains, very proud of his strategy.

Dawn draws the array and checks to see whether everyone understands by asking who can explain Daniel's strategy in their own words. Several students do so, and Dawn seems satisfied that the group understands. "That's a really neat strategy, isn't it?" Daniel beams, and Dawn turns to Ned, "And what did you do, Ned? Your hand was up so quickly. Did you see a relationship to another problem again?"

Ned laughs and says, "Yep. Just doubling and halving again. It's the same as 9×30. The 9 was halved and the 30 was doubled."

Several students exclaim in surprise. Dawn smiles and goes to the next problem: $2\frac{1}{2} \times 120$. This time everyone's hand is up quickly, and Dawn calls on Tanya, who has not yet spoken. Tanya, like the rest of the class, has made use of the doubling and halving relationships in this string of problems.

The other strategies that have previously been discussed are also powerful strategies, and Dawn does not want to imply that they should be replaced by doubling and halving. She is only trying to help her students think about relationships in problems, to look to the numbers first before calculating. To ensure that this happens, she follows with the next two problems: 15×36, then $15\frac{1}{2} \times 36$. Most students see the relationship between the first one and 30×18. Since they have already calculated 15×18, they know they just need to double that answer. A few students solve it by doing 10×36 to get 360, halving that to get 180, and then adding these partial products for an answer of 540. For $15\frac{1}{2} \times 36$ everyone uses the distributive property, adding 18 more for an answer of 558.

Dawn moves next to a more difficult problem, challenging her students to generalize: $5\frac{1}{4} \times 40$. She asks the students to write their strategy and solution in their math journal, and then to turn to the person sitting next to them and share it. Most of the students use the distributive property, but a few double and halve to make $10\frac{1}{2} \times 20$, and then double and halve again: $21 \times 10 = 210$. "Do you think there is any way we could have done something like that but with one step only? Could we eliminate the fraction in one step?" Dawn asks them to ponder with pair talk. After a few minutes, she starts the discussion: "Lynn?"

"Maybe we could times the $5\frac{1}{4}$ by 4, and then take a quarter of the 40?" Lynn offers somewhat tentatively.

Dawn writes, $5\frac{1}{4} \times 4 \times 10$. "Hmm...let's look at what Lynn is saying." Adding parentheses, she writes $(5\frac{1}{4} \times 4) \times 10 = 5\frac{1}{4} \times (4 \times 10)$.

Rich discussion ensues until a consensus is reached that Lynn's strategy works, and then Dawn asks the class to consider whether it might work for decimals, too. She writes the last problem in the string: 0.8×80. With elegant efficiency, many of the students write: $0.8 \times 10 \times 8 = 8 \times 8$. At the end of the minilesson, a new sign is made for the wall of strategies. Dawn asks the students to describe the strategy in their own words and they decide to write, "Group the factors in pretty ways."

These young mathematicians are composing and decomposing numbers flexibly as they multiply rational numbers. They are inventing their own strategies. They are looking for relationships among the problems. They are looking carefully at the numbers first before they decide on a strategy.

Students don't do this automatically. Dawn has developed this ability in her students by focusing on computation during minilessons, with strings of related problems every day. She has developed the big ideas and models through investigations, but once this understanding has been constructed, she promotes fluency with computation strategies in minilessons such as this one.

Using Models during Minilessons

As you work with the minilessons from this unit, you will want to use the related models to depict students' strategies. The money, clock, and double open number line models are helpful for addition and subtraction. Arrays and ratio tables are most helpful for multiplication and division. Representing computation strategies with mathematical models provides students with images of their strategies for discussion, and supports the development of the various strategies for computational fluency, but only if the models are understood. Modeling is developed through three stages. Initially models are introduced as a model of a realistic situation. In the second stage you can use it to model the computation strategies as students explain. Later, in stage three, it will become an important tool to think with.

Note: This unit assumes that the models have already been developed with realistic situations and rich investigations. In this series, the Field Trips and Fund-Raisers unit can be used to develop the double open number line (particularly the fund-raiser investigation in the second week); Exploring Parks and Playgrounds develops the array model for multiplication of fractions; and Best Buys, Ratios, and Rates develops the ratio table and provides further experiences with the double open number line. If your students do not have well developed understandings of these models, you may find it beneficial to use these units first. Representations like these give students a chance to discuss and envision each other's strategies. Eventually students become able to use these models as tools to think with, to model, prove and explore their ideas.

A Few Words of Caution

As you work with the minilessons in this resource book, it is very important to remember two things. First, honor students' strategies. Accept alternative solutions and explore why they work. Use the models to represent students' strategies and facilitate discussion and reflection on the strategies shared. Sample classroom episodes (titled "Inside One Classroom") are interspersed throughout the unit to help you anticipate what learners might say and do and to provide images of teachers and students at work. The intent is not to get all learners to use the same strategy at the end of the string. That would simply be discovery learning. The strings are crafted to support development, to encourage students to look to the numbers and to use a variety of strategies helpful for those numbers.

Secondly, do not use the string as a recipe that cannot be varied. You will need to be flexible. The strings are designed to encourage discussion and reflection on various strategies important for numeracy. Although the strings have been carefully crafted to support the development of these strategies, they are not foolproof: if the numbers in the strings are not sufficient to produce the results intended, you will need to insert additional problems, depending on your students' responses, to support them further to develop the intended ideas. For this reason, most of the strings are accompanied by a section titled "Behind the Numbers" describing the string's purpose and how the numbers were chosen. Being aware of the purpose of each string will guide you in determining what additional types of problems to add. The Behind the Numbers sections should also be helpful in developing your ability to craft your own strings. Strings are fun both to craft and to solve.

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