Building on Student Ideas: The Border Problem
(from Boaler & Humphries, *Connecting Mathematical Ideas*)

TEACHER: So, uh, what I thought, right here is a ten-by-ten grid and what I'd like you to do mentally is figure out without counting one by one, how many squares are in the colored-in portion? How many unit squares? And without talking, without counting one by one, without writing. . . . Wow. Lots of people have an idea. Could you talk to each other just about what you got? Can I please have your attention back? All right, who has, by the way, do we all want to say the answer?
STUDENTS: Yeah.
TEACHER: What is it?
STUDENTS: Thirty-six.
TEACHER: Thirty-six squares. Good. Did anyone think it was maybe forty first? . . . At first?
STUDENTS: Yeah (murmur).
TEACHER: Did anyone think it was maybe thirty-eight first? A few people, OK. And, so, when you talked it over in your group, those of you [who] thought forty, what were you thinking? All right, Stephanie, what were you thinking?
STEPHANIE: Uh, I was thinking that one side is ten and then there's four sides and times ten by four is forty.
TEACHER: OK. How many were thinking just like Stephanie? All right, what about the thirty-eight people, what were you thinking? Uh, Mindy.
MINDY: I was thinking about the top two are ten and so ten plus ten equals twenty, and then the other two to get were nine each, so that really makes eighteen, and twenty plus eighteen . . .
TEACHER: OK. All right. So now let's just see some different methods—you know how we like to gather different methods. Let's see some different methods for getting the thirty-six. So, let's see, let's have Sharmeen. Sharmeen, what's your method?
SHARMEEN: Well, Krysta started out with forty from ten times forty and then I subtracted four from that because there would be four squares overlapping, and so that got thirty-six.
TEACHER: Raise your hand if you understand how Sharmeen did it. And the four was for . . . ?
SHARMEEN: The four squares that overlap.
TEACHER: OK. And this four was for?
SHARMEEN: The four sides.
TEACHER: And this ten was for?
SHARMEEN: The ten squares on each side.
TEACHER: OK. Another way to do it? Colin.
COLIN: All right, how I did it was, I just put one side was obviously gonna be ten, so it's ten, and then I did like the bottom one and that was gonna . . .
TEACHER: Could you go up and show us? I think that might help us.
COLIN: Can I use like a pointer-thinger (unintelligible)?
TEACHER: The pointer's in the top drawer if you want to use it . . . top . . . right in the center . . . there it is.
COLIN: All right I found it. Anyway, so I, like, I know this side is ten so I just did like ten and then this one, this one's the overlapping one, so then this would be nine, then this one would be nine, too, because this one's overlapping. And then for this, one it'd be eight because these two, this one is being used by this one and this one is being used by that one.
TEACHER: Colin, I really like the way you showed visually what you did. How many understood Colin's method? Wow. Great. OK. Thank you. How about a different method? Wow. There's a lot. All right. Umm, let's see. It's hard to choose . . . Joe.
JOE: I, um, I added the top. . . .
TEACHER: You want to go up like Colin did?
JOE: I added, I know that this one, this is ten and so is that, so I added those two and at first I thought this was like nine right there until I thought about it for a while and that's eight and so I added those two and that's sixteen and I added it to twenty.
TEACHER: So Joseph, look at this and tell me if you think that represents what you did.
JOE: Yeah.
TEACHER: OK, and the ten and the ten stand for?
JOE: This one and . . . this row right here, how many boxes in that row and

how many in that row.
TEACHER: OK. How many unit squares are in the top and how many are in the bottom. And then where'd you get the eight?
JOE: From like this row, except it didn't have these two.
TEACHER: Ah. How many understand Joseph's method? All right, are there any other methods? Oh, lots of people, but how many had Joe's method that they wanted to explain? A lot of people. Can anybody think of another way to do it? Melissa.
MELISSA: OK. Um, since the whole square is one hundred units, 'cause ten times ten. Um, there's, OK, that and then there's eight . . . (goes up to the board) There's eight by eight that aren't shaded, so that's sixty-four, and you subtract sixty-four from a hundred.
TEACHER: And you got thirty-six.
MELISSA: Yeah.
TEACHER: What do you think of that method?
STUDENTS: Cool. Good.
TEACHER: OK, so, that would be, so Melissa though, help me though. You got ten times ten first?
MELISSA: Yeah.
TEACHER: And then you subtracted eight times eight.
LEO: Oh, I get it.
TEACHER: OK. Any other methods? Tina?
. . .
TEACHER: All right, good point. Um, any other methods that you think you could do it? 'Cause there's one more method that came up in period one. I forgot whose method it was. Zachary's. Zach Morris?
STUDENT: Yeah, I know Zach.
TEACHER: All right, here's what I'm going to do. This is backwards. Rather than having you guess what Zachary's method is, which doesn't make any sense at all, I'm going to show you Zachary's method; you see if you can see why it makes sense with the picture. OK. Here's what Zachary did and Dana, I'll call on you in a minute.
Four times eight, plus four. Why does that make sense with the picture?
STUDENTS: Oh, oh (murmuring).
TEACHER: Kayla, why does that make sense with the picture?
KAYLA: Um. Because there are four sides . . .
TEACHER: Want to go up?
KAYLA: No. Um, there are four sides. OK, I'll go up . . . OK. There's like four sides and um, there's, wait, OK. And there's eight right here, eight right here and eight right here, yeah, and that'd be times four and then it'd be plus four 'cause one, two, three, four. Yeah.
TEACHER: Um. Anyone want to comment to her, see if you agree with her? Kayla, you call on someone.
KAYLA: Stephanie.
STEPHANIE: OK. Um, for the one side, if you said that there were eight there so you'd have two left over in the corner and then if you put eight on each other side, there'd be one in each of the three corners, or the two corners, two other corners, so basically if you do eight times four, then you're gonna have four left; that's why you add four.
TEACHER: Is that the same as what you said?
KAYLA: Yeah. What I was trying to say.
TEACHER: I thought you said it very well. Just show us again where the eight is coming from, though, Kayla.
KAYLA: Well, like, these, this comes from eight, right here . . .
TEACHER: So this whole side length, but what's, what's the part that you're leaving out?
KAYLA: The corners.
TEACHER: Oh, the corners, OK . . .
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TEACHER: OK. Kay wanted to comment on Zach's method.
KAY: Zach's method, I think, is pretty much the same as Tina's only that it's eight times four and not nine times four.
TEACHER: What's the difference . . . I really like the fact that Kay is connecting two methods to each other. How [are] Tina's and Zach's methods alike and how are they different? Say it again, Kay, and then I'll call on somebody else.
KAY: Well, I think it's pretty much the same. Well, Zach's method, four times eight, plus four, is pretty much the same thing as Tina's method nine times

four.

Teacher: All right. And any other comments about the two methods? Sarah.

Sarah: Well, the only thing Tina did was she, um, well, because four times eight plus four is the exact same thing as nine times four only you're saying instead of having the ninth four be, instead of having . . . instead of . . . I don't know how to say it . . .

Teacher: I'm distracted, Mindy, by your hand being up right now. Please listen to Sarah.

Sarah: OK. Instead of having the ninth four being multiplied with the other eight fours, times four, you're, he added it after he multiplied four times eight.

Teacher: So Sarah is thinking about this numerically. She's really, she knows that these two are equal and she's saying what, like, what the person did first or second. So, another way I'd like you to think about it is what's Zach doing that Tina's not doing and what's Tina doing that Zach's not doing? How would those look in the picture? Uh, so, Krysta, go ahead.

Krysta: Um, well, I think, kind of it's what I was saying before that we just took the corners away. And he took the corners away and added what was left. . .

Teacher: OK, stop for just a second; so Zach took both of the corners off.

Krysta: All four of the corners off and I . . . and Tina just took, she didn't take any, she just kept the corners there but she counted the sides minus one corner for each side.

Teacher: You know, we have Tina here to defend herself. Is that right, Tina?

Tina: Ummm . . . Yeah . . .

Teacher: Kind of right. All right, I'm going to . . . Oh boy, um, uh, Sharmeen.

Sharmeen: I think my method is also kind of like Tina's and Zachary's because, well, I was counting the sides and then I subtracted four, but he didn't count the sides and he added four.

Teacher: Ah, OK. Other comments about how those methods are alike and different . . . ?

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Teacher: Here's what I'm going to ask you to do. I'm going to turn that off and I want you to visualize a square in your mind and use whatever method you want, do I want you to use whatever method you want? Yeah, I want you to use whatever method you want, but I want you to shrink the square in your mind, down to a six by six. And then use one of those methods, but instead of there being ten unit squares on one side, now there are six. Use one of those methods and see what you think the total number of squares . . . and let's keep our hands down. It's kind of intimidating to have people that think so fast . . . six by six . . .

OK. Now, let's just say that I don't care about how many there are. Let's say that what I care about is how people would do it. So, what I want to know is, and not Sharmeen but someone else, if the square, if the big square was a six by six, what would Sharmeen have done to get the total number of unit squares on the border?

Teacher: What they would do then is, they would say, well, "Is there a way to shorten this?" So, how could we shorten this without doing all the writing and still communicate Joe's method, so that we could understand it? Does anybody have any ideas about how we could do that? You do, Krysta—how?

Krysta: By, like, using algebra. Like make, um, the fifteen and fifteen like xs and the thirteen and thirteen ys and so that you could say, like, x plus x and then label the square x plus x and then put all in letters that . . . (inaudible).

Teacher: I really like Krysta's idea of taking some kind of a symbol for, um, like, and she's using a letter. Does anybody know what letters are called in algebra, what they're called? Sharmeen, do you know?

Sharmeen: Like what they're meant for?

Teacher: Well, actually, they have a name. The name doesn't really say everything about what they do but . . .

Students: Variables.

Student: I knew that!

Teacher: They're called variables. You've heard that word before? OK. So, let's, let's take Krysta's idea and work with it. Let's pick a symbol and let's, let's say, um, for . . . do you want to use x, Leo? You guys like x? All right.

Um, we . . . you could choose l for Leo, a for Anna, k for Krysta; so now what are we going to do?

STUDENTS: F for Flores, h for Mrs. Humphreys . . .

TEACHER: H for Ms. Humphreys. How about, um, well, I have an idea. All right, I have an idea. How about if everyone picks their own variable. So, let's all pick your own variable. I don't care what it is in the book but, let's just, we have to tell what it means. OK, I'm still waiting for quiet and . . . OK, so what we're going to do is we're going to say "Let" and you get to pick your own variable and I'll pick, uh, I don't want to pick one, so, "Let 'something' equal . . . " and this is the important part . . . OK. What I'd like to be able to do is talk without being interrupted. . . . Up here this first sentence is the most important thing because it says what you need to know, what Joe needs to know to figure out the number of squares. The only thing Joe needs to know is how many unit squares are on one side. That's all he needs to know. So what we want to do is we want to let our symbol equal the number of unit squares on one side. And I guess I will, I'm just going to use x because it seems strange to leave it blank. But you all can choose whatever one you want, but it's really important that you write this part out. "Let x equal the number of unit squares on one side"—and of course you might not have x. Now here's what I'd like you to try to do in your tables. I'd like you to try to translate this into an algebra expression. Like Joe is telling us what to do. Here's a picture of what Joe said to do. Here are some examples of what Joe said, and before we do let's just look at these for a minute. What's staying the same in this arithmetic? Pam, what's staying the same?

PAM: Um, the like, the, you're always adding. You're not, I mean, you could multiply, but you're always adding.

TEACHER: OK. Anything else change, uh change—staying the same? And, uh, what is changing? Pam, I mean, Sarah.

SARAH: Well, the first two numbers are the same numbers and the last two numbers are the same numbers.

TEACHER: OK, so these two are the same, these two are the same, but those are different. So, do you think you could look at this, look at this and look at this and write an algebraic expression? Why don't you try at your table? (students talk in small groups).

SHARMEEN: S and so $s + s + (s - 2) + (s - 2)$. Though that's kind of complicated. Is there any other way to put it?

ANTONY: What is it?

SHARMEEN: Um, mine? Was $s+ s+ (s - 2) + (s - 2)$.

KIM: No we had to, like, um, how about we write a variable for . . . (inaudible) . . . make a variable for thirteen.

SHARMEEN: Yeah, oops. Oh, m equals . . . OK, so it's $s+ s+ m+ m$.

TEACHER: (Addressing whole class) OK. May I please have your attention at this table for a minute? Do you want to say the theory, your theory, about the other letter?

PAM: OK. Well, um, we knew that we had to add, like, the top and the bottom or up there would be the green and we knew that that would be our, just say our variable is m, so you'd have to add m plus m, but we're trying to figure out a way of, um, subtracting two without saying subtracting two from the sum of m plus m. So we figure that we need, um, another letter for the two sides but we're having a hard time figuring out how to say it.

TEACHER: Why don't you check and see if anybody has any ideas for you.

STUDENT: I don't understand.

TEACHER: Oh, wait a second. Does everyone understand her, the issue at this table?

STUDENTS: No.

TEACHER: Oh. Travis, what's the issue at this table?

TRAVIS: It's, it's uh, . . . I had it . . .

PAM: OK, so we're adding the top and the bottom but we're trying to, uh, figure out how to add the sides but we know we have to subtract two and, we don't know, you can't just say the top plus the bottom minus two because that would mean that you'd be subtracting it from the sum of the top and bottom. So we're trying to figure out, we know we have to use, we think we have to use another letter but we're having a hard time figuring that out; what, like, how to say subtracting two with the other letters.

TEACHER: So since Pam went to all that trouble of explaining, Mindy or Kayla or Joe, would you choose someone in the class to help you out, help give you ideas?

JOE: Stephanie.
STEPHANIE: Um, I think maybe you could, like, do x plus x for the top and the bottom, or whatever letter you're using, and then you could do (x – 2) + (x – 2). And that would give you the, um, the total border.
STUDENT: And then you wouldn't have to do another letter?
STEPHANIE: Yeah, and then you could keep the same letter and you're still taking away two.
PAM: Oh, OK.
TEACHER: Why does that work? Why does that make sense? . . . Melissa, why does that make sense?
MELISSA: All the sides are the same length, so you only need one letter and you have to subtract two.
TEACHER: OK. That whole thing about when, why you might need another letter. Kimberly, you were thinking that you need another letter, right? Why were you thinking that? Because that's a really important thing; when do you need another letter and when don't you? Why did you think you did?
KIMBERLY: Because we have, um, four letters and I was thinking, cause she read her, um, her thing there and I said it was kind of complicated so I was thinking that (inaudible) . . .
TEACHER: OK. What do you think now?
KIMBERLY: We don't need it anymore.
TEACHER: You're sure? You're convinced? OK. Um, Travis, were you going to add anything to that? OK. Yeah, Pam.
PAM: Well, the reason I was thinking we needed another letter is because in the beginning we needed two different numbers. So, maybe you needed two different letters.
TEACHER: Oh. Oh, that's . . . yeah, right, right.
PAM: It kind of, like, if you were talking about what's the same and what's different. So in the algebraic formula it's, the things that are the same and different are not the same and different on the numbers.
TEACHER: Right.
PAM: So, I think that when you asked what was the same and what was different, it kind of confused me.
TEACHER: I'm glad it confused you in that way because it brought out a really important thing that we're going to be grappling with in some other problems. Because sometimes you are going to need a different letter, and sometimes you're not. Why don't you need another letter in this case? I mean, like, if you can do it without another letter, you want to keep it simpler. Why can we, why can we? Sarah. Yeah, this Sarah. Sarah Stanley.
SARAH: Because you can do, you can do things to that, uh, letter, to make it the number that you want without using a different letter for that number.
TEACHER: And in this case the thing that you're talking about doing is . . .
SARAH: I said, well, I used s as my, um, . . .
TEACHER: OK, so let's write down s; I'll write down s for Sarah.
SARAH: OK. And, so I said, in parentheses, I said s times two.
TEACHER: Now Sarah, I'm going to stop you here because I agree with you that s time two is the same as s plus s, but I want to keep it as consistent with the way Joe is seeing it as possible, so I'm just going to say s plus s.
SARAH: OK.
TEACHER: OK.
SARAH: And then, like, that's in parentheses, and then I, outside of the parentheses, I put + (s x 2 – 4).
TEACHER: Oh . . .
PAM: But that's a different method.
TEACHER: OK. So maybe that's a different method. Can you think about what Joe would actually do and actually that's another method I want to come back to. That's interesting. So, Sarah, and actually, I'm going to write this down. Sarah wrote, s times two, Sarah, and then you wrote, "+ (s x2 – 4)." We're going to have to come back to that. I don't want to lose it so I'll save this transparency. But what would Joe have done exactly? Travis.
TRAVIS: S plus sand then plus s minus two.
TEACHER: OK.
TRAVIS: And then plus s minus two.
TEACHER: Um, let's see. Who would read their sentence to us? I know the bell is about to ring but would anybody . . . oh darn . . . OK. Think about this tonight and tomorrow we'll get to work on it some more. Tonight's homework is . . .