

Developing Children's Understanding of the Rational Numbers: A New Model and an Experimental Curriculum

Joan Moss and Robbie Case, *University of Toronto, Canada*

A new curriculum to introduce rational numbers was devised, using developmental theory as a guide. The 1st topic in the curriculum was percent in a linear-measurement context, in which halving as a computational strategy was emphasized. Two-place decimals were introduced next, followed by 3- and 1-place decimals. Fractional notation was introduced last, as an alternative form for representing decimals. Sixteen 4th-grade students received the experimental curriculum. Thirteen carefully matched control students received a traditional curriculum. After instruction, students in the treatment group showed a deeper understanding of rational numbers than those in the control group, showed less reliance on whole number strategies when solving novel problems, and made more frequent reference to proportional concepts in justifying their answers. No differences were found in conventional computation between the 2 groups.

Key words: Children's strategies; Cognitive theory; Curriculum; Decimals; Elementary, K-8; Fractions; Instructional intervention; Ratio, proportion, percent; Rational number/representations

The domain of rational numbers has traditionally been a difficult one for middle school students to master. Although most students eventually learn the specific algorithms that they are taught, their general conceptual knowledge often remains remarkably deficient. In the domain of fractions, for example, the majority of Grade 9 students, when asked to estimate the sum of $11/12 + 7/8$, choose 19 or 20 as the answer in a multiple choice format (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980). In the domain of decimals, most middle school graduates assert that "large" numbers such as 0.1814 are bigger than "small" numbers such as .3 or .385 (Hiebert & Wearne, 1986). Percents appear to be no easier: When asked to compute 65% of 160, the majority of high school students either fail to give any answer at all or give answers that are off by more than an order of magnitude (e.g., 2.5) (see Moss, 1997). Although the foregoing errors

The research reported in this article was supported by grants from the Social Sciences and Humanities Research Council of Canada and the J. S. McDonnell Foundation. We would like to express our thanks to the three anonymous reviewers for their comments on an earlier version of the manuscript, to Janette Pelletier for her help in developing the materials that were used in the curriculum, and to Cheryl Zimmerman and Barbara Mainguy for their help in typing the manuscript.

are quite diverse, they all reveal a profound lack of conceptual understanding that extends across all three rational number symbolic representations and calls our existing methods of teaching these representations into serious question.

Several explanations have been proposed for the difficulties that students experience in mastering the rational numbers with current teaching methods:

1. *Syntactic versus semantic emphasis in training.* The first explanation is that in middle school mathematics programs too much time is devoted to teaching procedures for manipulating rational numbers and too little time to teaching their conceptual meaning; in effect, syntactic knowledge is given precedence over semantic knowledge (Hiebert & Wearne, 1986; Resnick, 1982).
2. *Adult- versus child-centered instruction.* A second explanation is that teachers take no account of children's spontaneous attempts to make sense of the rational numbers, thus discouraging children from attempting to understand these numbers on their own and encouraging them to adopt an approach based on the rote application of rules (Confrey, 1994; Kieren, 1992; Mack, 1993).
3. *Use of representations in which rational and whole numbers are easily confused.* A third explanation is that, when attempts are made to emphasize meaning in introducing rational numbers, rational numbers are not sufficiently differentiated from whole numbers. A particular problem that has been cited in this regard is the use of pie charts as vehicles for introducing children to fractions (Kerslake, 1986; Kieren, 1995; Mack, 1990; Nunes & Bryant, 1996; Ohlsson, 1988).
4. *Problems with notation.* A fourth explanation is that in most middle school mathematics programs rational number notation is treated as something that is transparent (i.e., that can simply be given by definition at the outset of a lesson). In fact, rational number notation—particularly the notation for decimals—entails significant problems in its own right. By ignoring these problems, instructors once again make it harder for children to make sense of the underlying conceptual system (Hiebert, 1992).

The above explanations are not mutually exclusive, of course. Still, each one does suggest a somewhat different solution to the problem of improving the teaching of rational numbers. Depending on which explanation one sees as most promising, one may be inclined to pursue a particular sort of curricular reform. For example, Hiebert and Wearne, in their analysis of children's difficulties in mastering decimal fractions (1986), focused on the first and fourth of these explanations. In keeping with their analysis, they devised a remedial program in which base-ten blocks were used as props for teaching decimals, and they encouraged children to represent problems with these blocks before attempting to solve the problems (Hiebert, Wearne, & Taber, 1991). In explaining children's difficulties in learning fractions, Kieren (1994a) focused primarily on the second and third explanations. In keeping with this analysis, he developed a program for teaching fractions that used paper folding rather than pie cutting as its primary

problem situation. Finally, Streefland (1991, 1993) focused on all four explanations. In his procedure for teaching fractions, children were presented realistic sharing situations (e.g., five children sharing two pizzas). To represent these situations, children used a notation that they devised themselves; the notation emphasized proportional rather than additive relations.

All the foregoing programs achieved at least some degree of success, thus demonstrating that—with revised conditions of instruction—children can be led to a deeper understanding of some aspects of the rational number system. Still, there is a growing concern that researchers may need to attack the problem in a broader and more integrated fashion. In their recommendations for curriculum reform, Post, Cramer, Behr, Lesh, and Harel (1993) suggested that curriculum developers' attention should be directed away from the attainment of individual tasks toward the development of more global cognitive processes. A similar point has recently been made by Sowder (1995) and by Markovits and Sowder (1991), who suggested that children need to learn how to move among the various possible representations of rational number in a flexible manner. Although they retain a concern for deep conceptual understanding, contemporary analysts are clearly urging us to create curricula that will help children develop better overall conceptions of the rational number system as a whole and the way its various components fit together—not just better understandings of one or another of these components in isolation.

For the project that we describe in this article we had this general goal as the overriding objective. In this article, we first outline the general theory on which the project was based; we then describe the curriculum that we established and the results to which it led.

THEORETICAL FRAMEWORK

As Kieren (1994b) has pointed out, researchers who have undertaken theoretical analyses in the domain of rational numbers have adopted two general stances: epistemological and psychological. Those with an epistemological thrust have been devoted to clarifying (a) the nature of rational numbers as mathematical constructs and (b) the subconstructs of which they are comprised (e.g., ratio, quotient, measure, and operator). Those with a psychological thrust have been devoted to identifying the schemas that children bring to the domain of rational numbers and the way in which these schemas develop when the children are introduced to the domain in a more formal fashion.

Our study had its origins in the psychological tradition. The basic hypothesis on which the study was premised was one proposed independently by Case (1985) and by Resnick and Singer (1993): Children's understanding of whole and rational number develops in a formally similar way. In each case, children's numerical and global quantitative schemas develop separately at the outset. While they make the transition to a higher level of thought, children gradually coordinate these two schemas to yield a core understanding both of the way in

which the simplest numbers in the field in question are structured and of the notation that is used for representing them. This core understanding is then extended to more complex numbers and forms of representation until the overall structure of the entire field is understood.

According to this analysis, the differences between the development of whole number and rational number understanding are the nature of the original psychological units and the stage of development at which the psychological units are first integrated into a single schema. For whole numbers the two “primitive” psychological units are (a) the schema for verbal counting and (b) the schema for global quantity comparison. At about age 6, coordination of these two structures yields a “mental counting sequence” that children then learn to notate in a conventional fashion. When they grow older and receive further instruction, children gradually learn to count by tens and hundreds as well as by ones and to understand the relationships among the different units that are created in this fashion. By the age of 10, children have assembled a generalized understanding of the entire base-ten system and of the form of notation that is used for representing it (Griffin & Case, 1996; Okamoto & Case, 1996).

For rational numbers the two primitive psychological units are (a) a global structure for proportional evaluation (Noelting, 1980a, 1980b; Resnick & Singer, 1993) and (b) a numerical structure for “splitting” or “doubling” (Case, 1985; Confrey, 1994; Kieren, 1992), both of which appear to be in place by about age 9 to 10 years. Coordination of these two structures at the age of 11 to 12 yields the first semiabstract understanding of relative proportion and simple fractions (especially $1/2$ and $1/4$). When children grow older and receive further instruction, they learn about different forms of splits and the relationships among different sorts of fractions. They also learn about the relationship between fractional and decimal notation. Eventually (though often not until they have reached the end of high school) they construct a generalized understanding of the entire rational number system.

According to this analysis, one of the most important roles that instruction can play is to refine and extend the naturally occurring process whereby new schemas are first constructed out of old ones, then gradually differentiated and integrated. In the present project, therefore, we selected children who were just entering the second stage in their development (i.e., children who were 10 to 11 years of age) and who had the prerequisite schemas already in place. We then designed a curriculum that we hoped would help them integrate their existing understandings in a natural fashion and use the resulting cognitive structure as a basis for understanding the overall structure of the rational number system.

Although it has been suggested that there is a necessary order in which the different forms of rational number representations must be learned, our own intuition is closer to that of Confrey (1994), who believed that the order is more arbitrary and that what matters is that the general sequence of coordinations remains progressive and closely in tune with children’s original understandings. We also share Mack’s (1990) intuition that the teaching of one form of representation in

some depth is preferable to the superficial teaching of several different forms at once. In keeping with these two intuitions, we used a single form of representation and we presented children with the sequence of tasks that we felt would maximize the connection between their original, intuitive understanding of ratios and their procedures for splitting numbers. We then gradually introduced the other forms of representation and their relationships to the original one. Our general procedure was as follows:

1. The visual prop that we selected at the outset was a beaker of water. There is substantial psychophysical literature that indicates that humans have little trouble either in seeing such objects in global, proportional terms (i.e., as being full, nearly full, about half full, nearly empty, or empty) or in selecting proportional rather than absolute matches for them. There is also considerable developmental literature that indicates that by age 10 to 11 children have a well-developed sense of the whole numbers from 1 to 100. Accordingly, our first exercises were ones in which children were asked to assign a numerical value from 1 to 100 to water levels in various beakers in order to estimate their “fullness.” In short, for these first exercises children were asked to think about relative height in terms of fullness and percent.
2. While the lessons were progressing, we encouraged children to coordinate their intuitive understandings of percents in this context with their strategies for manipulating the numbers from 1 to 100. The two strategies that we emphasized were *numerical halving* (100, 50, 25, etc.), which corresponds to a sequence of visual-motor splits, and *composition* (e.g., $100 = 75 + 25$), which corresponds to visual-motor addition of the results.
3. Once children understood how percentage values could be computed numerically, in a fashion that corresponded directly to intuitively based visual-motor operations, our next step was to introduce them to two-place decimals. We did so in a measurement context by explaining that a two-place decimal number indicates the percentage of the way between two adjacent whole number distances that an intermediate point lies (e.g., 5.25 is a distance that is 25% of the way between 5 and 6). We then gradually expanded this original idea to include multiplace decimals, using a transitional “double decimal notation” that the children spontaneously invented (e.g., 5.25.25 is a number that lies 25% of the way between 5.25 and 5.26).
4. Finally, we presented children with exercises in which fractions, decimals, and percents were to be used interchangeably.

The psychological structure that we hoped children would construct as a result of the sequence above is illustrated in Figure 1. The top line of the figure illustrates the perceptually based sequence of ratios that we hoped children would learn to recognize and to order at the outset of the program. The (left to right) arrows connecting the icons in this row indicate the operation by which we presumed children would move from one element to the next in the sequence. This operation, which might best be termed *visual-motor halving*, is most easily executed by putting

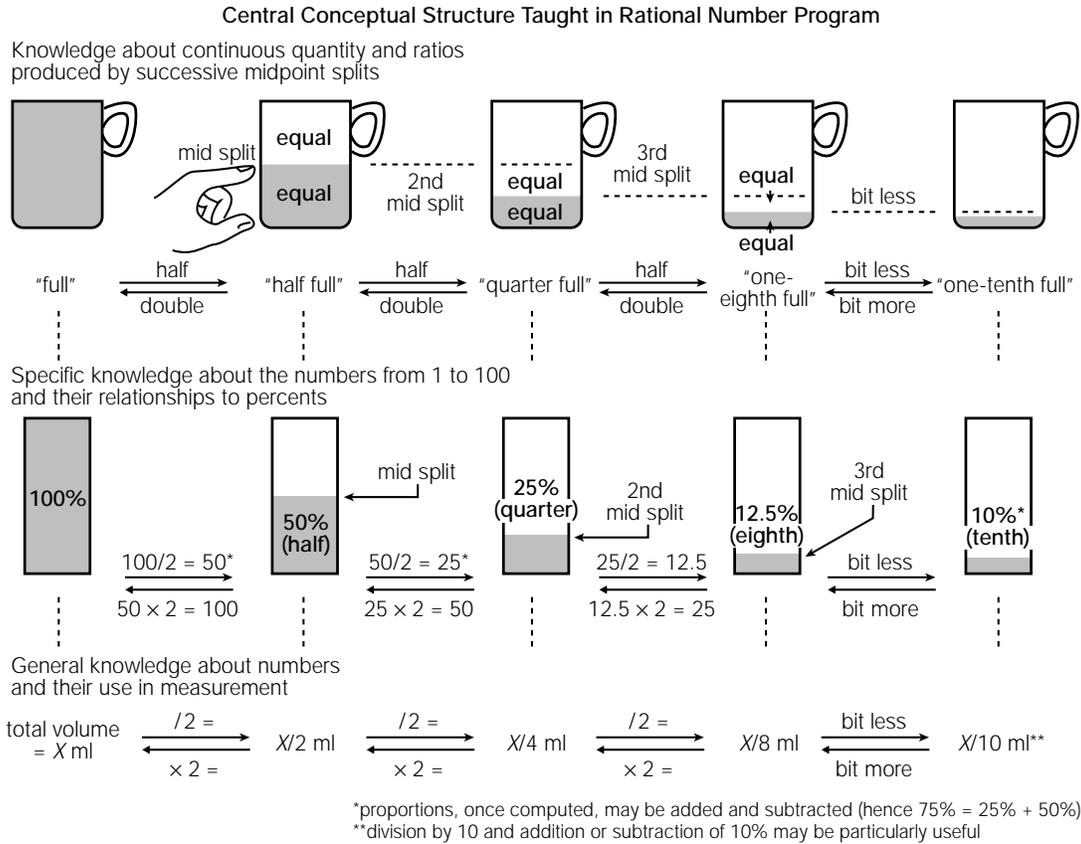


Figure 1. Central conceptual structure taught in the rational number program.

one’s forefinger beside an object—then moving it up and down until one finds the point at which the top and bottom halves of the object are symmetrical.

In the second line of the figure appear the corresponding representations that we hoped children would develop for benchmark percentage values and the numerical operations that connect the representations. Once again, we presumed that children would start with 100% and then calculate half of this value for each successive visual-motor split. We also presumed that children could learn to compose and decompose percentages that were calculated in this fashion (e.g., to determine the size of 75% by finding the sizes of 50% and 25% and then combining them).

Finally, the bottom row of the figure is meant to represent the corresponding set of measurement techniques and formal arithmetic procedures that we hoped children would learn to use when the goal was to express a ratio in some standard set of units such as milliliters. For example, if one knows that the total volume a beaker can hold is 120 ml, one can determine what 75% of that volume must be by first computing half of 120 (60), then computing half of the resulting total (30), then adding these two values.

One can think about the psychological structure represented in Figure 1 as containing a rich network of icons, symbols, and procedures that children can access

and apply in a flexible fashion to insure that both their qualitative procedures for assessing and transforming continuous quantities and their more formal, arithmetic procedures will remain closely integrated. In this sense, the structure is directly parallel to the central conceptual structure for whole number that has been analyzed by Griffin and Case (1996): Within the whole number structure, children learn to integrate their qualitative procedures for evaluating and transforming discrete quantities with their more formal arithmetic procedures for addition and subtraction.

We reasoned that once children possess a ratio-measurement structure such as that diagrammed in Figure 1, they should be able to use this structure as a starting point for learning about decimals and fractions. The details of this sequence are described in the next section. Before passing to that section, however, we would like to emphasize that—in introducing children to the rational number domain in the order described (percents, decimals, then fractions)—we were aware that we were reversing the normal order for introducing these different representations. Our decision to do so was based on the following considerations:

1. By the age of 10 or 11, children have well-developed qualitative intuitions regarding proportions (Noelting, 1980a); they also have well-developed intuitions about the numbers from 1 to 100 (Okamoto & Case, 1996). By beginning with percents, we allowed them to bring these two sets of intuitions together in a natural *unidimensional* fashion.
2. By beginning with percents, we were able to let children use a form of visual representation with which they were already familiar, namely, the “number ribbon” that is used on the Macintosh computer when a file is being transferred (see Figure 2).¹ This representation further contributed to building a solid connection between children’s intuitions about proportions and their intuitions about numbers.
3. By beginning with percents rather than fractions or decimals, we postponed the problem of having to compare or manipulate ratios with different denom-

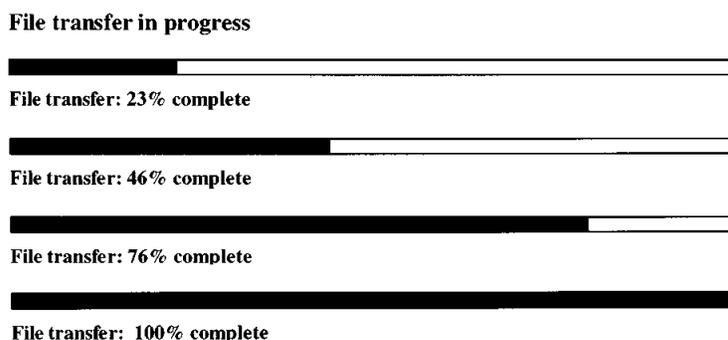


Figure 2. The “number ribbon” used on the Macintosh computer when a file is being transferred.

¹ All the computers that the children had access to in school were Macintosh machines, and thus this icon was already a familiar one.

- inators, thus allowing children to concentrate on developing their own procedures for comparison and calculation rather than requiring them to struggle to master a complex set of algorithms or procedures that might seem foreign to them.
4. Every percentage value has a corresponding fractional or decimal equivalent that is easy to determine. The converse, however, is not true. Simple fractions such as $\frac{1}{3}$ and $\frac{1}{7}$ have no easily calculated equivalent as percentages or decimals. By beginning with percents, we allowed children to make their first conversions among the different systems in a direct and intuitive fashion and thus to develop a better general understanding of how the three systems are related.
 5. Finally, although this reason was not central to our decision, we noted that the children appeared already to know a good deal about percents from their everyday experiences (Parker & Leinhardt, 1995). Before we began the instruction, we asked the children if they had ever heard percentage terminology used in their homes or daily lives. Not only were they able to volunteer a number of different contexts in which percentages appeared (their siblings' school marks, price reductions in stores having sales, and tax on restaurant bills were the ones most frequently mentioned), they were able to indicate a good, qualitative understanding of what different numerical values "meant," for example, that 100% meant "everything," 99% meant "almost everything," 50% meant "exactly half," and 1% meant "almost nothing." By beginning with percents rather than fractions or decimals, we were able to capitalize on children's preexisting knowledge regarding the meanings of these numbers and the contexts in which they are important (see Lembke & Reys, 1994, for further discussion on this point).

METHOD

Drawing on the foregoing analysis, we developed an experimental curriculum for teaching rational numbers, a curriculum based on intuitive estimation of ratios and their representation via number-ribbon diagrams and percentages. We then conducted a training study in which children who were exposed to our experimental curriculum were compared with children exposed to instruction of a more classic nature. To compare the two groups, we designed a detailed interview (the Rational Number Test) to assess children's conceptual understanding of fractions, decimals, and percents and of the relationships among them.

Participants

Twenty-nine fourth-grade students participated in this study. The experimental group was comprised of 16 fourth-grade students from a laboratory school located at the University of Toronto. The comparison group was comprised of 13 fourth-grade students from a private school that served a similar population.² The

² In each group, four students were excluded from the original cohort because the teacher judged that their time would be better spent reviewing more elementary concepts. The numbers listed are for students who actually participated in the study, not for the total class.

second school was located in the same general neighborhood as the first. It was also founded on similar principles and shared many features with the first school, including strong academic programs, small classes, individualized attention, a pedagogical style that was child-centered, and a strong commitment to quality instruction in mathematics. Finally, the students from the two classes were well matched on mathematical ability as measured by a standardized test of mathematics achievement, the Canadian Test of Basic Skills. The percentile score on the Concepts and Computation subscales of this test was 75 for both groups.

The curricula to which the children had been exposed prior to this study were also very similar. Both groups had been taught from the same mathematics text and had followed the same general curricular sequence, with one exception. In preparation for this study, the Grade 3 teacher for the experimental group had omitted the short unit on fractions and decimals that the text includes; instead, she had substituted three pilot lessons on percents that we designed (for more information on this point, see Moss, 1997). At the outset of the experiment, then, the two groups were very similar, although the experimental group had received a slightly different (and, we would argue, sounder) preparation for rational number work and thus knew a little more about percents.

Design

The Rational Number Test was administered immediately before and after the instruction to both groups. To control for experimenter bias, half the interviews were administered by one of the researchers (Joan Moss), the other half by a research assistant who was uninformed as to the purpose of the study.

The experimental group received twenty 40-minute instructional sessions spread over a 5-month period. These sessions were taught by one of the researchers (Joan Moss) and were presented at the rate of one lesson per week during the regular mathematics period. The control group received twenty-five 40-minute lessons spread across a slightly shorter time interval, again during the regularly allotted period for mathematics. Both classroom teachers continued to use the same text for the part of the curriculum that did not deal with rational numbers, and both attempted to limit their work with rational numbers to the periods set aside for that purpose. Both teachers also continued to use the same general style: one that drew heavily on manipulatives and made considerable use of group work and discussion. Thus, the main difference between the two groups was in the curriculum that they received for their work with rational numbers.

Experimental Curriculum

The experimental curriculum began with exercises in which children used percentage terminology to describe the fullness of different containers of water (“Approximately what percentage of this beaker do you think is full?”) or to guess the level of liquid in a container filled to a particular percentage value (“Where will the liquid come to in this beaker when it is 25% full?”). Two sets of props were used for the initial lessons. One was a set of glass beakers (differ-

ing only in their heights) that could be filled with water. Another was a set of large black drainage pipes with white venting tubes on the outside that could be raised from the bottom to different heights or lowered (thus simulating the action of water filling them to different levels). Both sets of props provided a “side view” that could be represented by a narrow vertical rectangle with some portion from the bottom shaded. To illustrate the problems that they had solved, students drew diagrams of such rectangles in their notebooks.

As it turned out, children’s natural tendency when confronted with the fullness problems was to use a *halving* strategy, that is, to determine where a line representing 50% would go on the cylinder or rectangle, then 25%, then 12.5%, and so on. This strategy was encouraged whenever possible. In the course of the instruction, we noticed that in figuring out the location for the 75% mark on a container, children spontaneously decomposed this value into 50% and 25%, then figured out each of these quantities by halving, then added the resultant amounts. This strategy, too, was encouraged.

When we introduced numerical problems for which precise calculations had to be made (for which children had to compute, for example, the amount of liquid that would be required to fill a 900-ml bottle 75% full), the children spontaneously used these same strategies. That is to say, they spontaneously began by calculating 50% of 900 ml (450 ml) and 50% of 450 ml (225 ml) and then added these two values. To facilitate this process, we initially presented problems that could be solved precisely using this general strategy. Problems that could be solved by calculating 10% or some multiple of 10% were introduced next, in the contexts of menus, tips, and tax.

In attempting to solve such problems, students began with a strategy that used halving to calculate a precise value, 12.5%, and then estimated 10% as “a slightly smaller number.” Subsequently, to compute more precisely, several students began to draw on their knowledge about money. For example, one student said that because one dollar has 100 cents, then 10% of one dollar is 10¢ and 10% of 200 is 20. When problems such as this were discussed, the conventional 10% strategy (i.e., divide by 10) gradually became established as an alternative to the halving strategy for certain kinds of questions.

Once children were comfortable in solving problems of both sorts, we introduced them to two-place decimals, using percents as the entrée. Large laminated number lines—with each number set exactly 1 m from the previous one—were placed on the classroom floor. Students were asked to walk some percentage of the distance between two adjacent numbers; it was explained that the total distance they had traveled could be represented with a two-place decimal number in which the whole number represented the number of meters walked and the decimal number represented the percentage of the distance to the next meter mark (e.g., “When you pass the 2-m mark and walk 75% of the way to the 3-m mark, the point you reach can be written as 2.75 m.”).

Although the foregoing context was an arbitrary one, children seemed to understand it immediately and were eager to apply their preexisting knowledge

of percents to decimal numbers. The exploration of decimals continued with the use of LCD stopwatches with screens that displayed seconds and centiseconds.³ These values, which appeared on the screen as decimal numbers, were interpreted as temporal analogs of distance, that is, as numbers that indicated the percentage of time that had passed between any two whole-second values. A number of exercises and games were presented to build up children's intuitive sense of small time intervals and their facility in representing these intervals in this format (e.g., "Try to start and stop the watch as quickly as possible, three times in a row. What was your shortest time? How does this compare to the shortest times in your group?"). Because the shortest times were often in the range of 9 to 15 centiseconds, these problems presented children with frequent opportunities for meaningful comparison of numbers that the literature suggests they tend to misrepresent, such as .09 and .15. The exercises were extended so that these time intervals were represented as percents, decimals, and fractions of a second.

The number-line exercises were continued through the playing of board games in which the children had to make moves forward or backward along a route consisting of 20 individually laminated 10-cm number lines. Each number line was calibrated in tenths and hundredths. The distance and direction that a child moved on this board was determined by his or her drawing two cards for each move: a number card with two digits and a direction card with either an addition or a subtraction symbol. Children had to determine the direction for the move on the basis of whether they had received an addition or a subtraction card, then insert a decimal point and a zero somewhere on the number card, and finally move along the board the stipulated distance. To succeed at this game, students were constantly required to construct decimal numbers that would correspond to a particular distance and to think about whether they wanted this distance to be a long or a short one.

After children had played this game once or twice, we suggested that they create their own board games with the goal of improving their facility with decimal notation still further. In the course of constructing these games they appeared to reflect on the strengths and weaknesses in their understandings of decimal notation and to consider optimal strategies that various game situations would encourage.

A special word must be said about fractions. Fraction terminology was used throughout the program, but only in relation to percents and decimals. At the beginning, all the children naturally used the term *one half* interchangeably with 50% and most knew that 25% (the next split) could be expressed as *one quarter*. We also told them that the 12 1/2% split was called *one eighth* and showed the children the fraction symbol $1/8$. In the final lessons of the experimental instructional sequence, we had a lesson in which we made fractions the focus. After that lesson, students were involved in solving and posing various challenges with

³ As it turned out, children were already familiar with the term *millisecond*; thus, it was explained that milliseconds involve a finer calibration of a single second, into 1000 instead of 100 units.

mixed representations. These activities included (a) *true or false exercises* such as “0.375 is equal to $\frac{3}{8}$, true or false?” (b) *stopwatch games* with instructions like “Stop the watch (the LCD stopwatches described above) as close to the sum of $(\frac{1}{2} + \frac{3}{4})$ as possible, and then figure out the decimal value for how close you are”; and (c) *Challenge Addition* in which students were asked to invent a long, mixed-addition problem, such as $\frac{1}{4} + 25\% + 0.0625 + \frac{1}{16}$, with which to challenge their classmates (or the teacher, an even more popular activity). A complete list of both the content covered in each lesson and the challenge problems that were assigned was presented by Moss (1997).

Control Curriculum

The control group devoted a slightly longer time to the study of rational numbers but followed the program from a widely used Canadian mathematics text series. The first topic in the text was fractions, which were defined as numbers that describe parts of a whole and which were illustrated with pie-chart diagrams. Exercises followed in which children were to determine fractions of a set, compare different fractions with regard to magnitude, and determine equivalent fractions. Decimals were taught next, using pie graphs, number lines, and place value charts. Tenths were introduced first, and their relation to single-place decimals was shown. Finally, equivalent decimals were taught by showing that numbers such as 0.3 and 0.30 are merely alternate representations of $\frac{3}{10}$ and $\frac{30}{100}$. Lessons involving operations with decimals were introduced next. The rules for addition and subtraction of decimals, as well as for multiplication of one- and two-place decimals, were taught explicitly, with careful attention to the significance of place value. The use of a fraction as an operator and computations involving division of decimals were taught at the end of the sequence.

Large- and small-group activities were used in the control classroom as well as in the experimental classroom, and in both settings, discussions were held and a wide variety of manipulatives and games were introduced.

Assessment Interview

In the interview, children were presented with 41 items in the pretest interview and 45 in the posttest interview—the original 41 and 4 new items. The test was ordered as follows: 12 percent items, 13 fractions items, and finally 16 items that featured decimal questions. The posttest interview was expanded to include one more percent item and three more items that featured fractions or decimals. To analyze the data, we assigned all but four of the questions to six subcategories:⁴ (a) *nonstandard computation* (e.g., Another student told me that 7 is $\frac{3}{4}$ of 10. Is it?), (b) *compare and order* (e.g., Which is bigger, 0.20 or 0.089?), (c) *misleading appearance* (e.g., Shade $\frac{3}{4}$ of this pie [already portioned into eighths,

⁴ Four of the early items in the test were very general in nature and were thus not assigned to a particular subcategory.

not fourths]), (d) *word problems* (e.g., A CD is on sale. It has been marked down from \$8.00 to \$7.20. What is the discount as a percentage of the original price?), (e) *interchangeability of representations* (e.g., What is $1/8$ as a decimal?), and (f) *standard computation* (e.g., What is $.5 + .38$?). (For the complete pretest and posttest interviews, see Moss, 1997.)

Many of the questions were selected from the existing literature on mathematics education, but several new questions were constructed as well. In assembling the overall battery, we intentionally included a number of questions that were close in their content to the sort of training that the experimental group received and a number that were closer to the training received by the control group. At the beginning of the test, for example, children were asked to find 25% of 80 ml, which was very similar to the sort of problem included at the beginning of the experimental program. Toward the middle of the test, students were asked to shade various fractional parts of a circle, a problem similar to the sort that had been presented in the control program. All but three of the items were scored dichotomously so that there was one point given for a correctly computed answer and no score given for an incorrect response.⁵

RESULTS

A preliminary analysis was run to determine whether there was any systematic tester bias. Because none was found, the results from both tester groups were combined in all subsequent analyses. Table 1 shows the pretest and posttest means for the two groups on the 41 pretest and 45 posttest items in the Rational Number Test. As may be seen, both groups showed some improvement, but the improvement of the treatment group was greater. A two-way analysis of variance with repeated measures was conducted to assess the significance of this pattern. The results showed a strong treatment by pre-post interaction in the predicted direction ($F(1, 32) = 29.06, p < .001$). We found a similar interaction when we analyzed the results for the decimals, fractions, and percent questions separately. Such interactions were also found for five of the six subcategories of test items, as indicated in the following sections.

Nonstandard Computation

On the items requiring nonstandard computation, the treatment group achieved a mean score at posttest of 6.78 (out of 9) as compared to 2.76 for the control

⁵ The item “What is 65% of 160?” was scored as correct if a student provided a reasonable calculation strategy as well as a numerical answer that deviated by no more than 4 from the correct answer of 104 (i.e., 100 to 108). The second item that had a special scoring criterion was the question “What is more, .06 of $1/10$ or .6 of $1/100$?” Students were given a passing score on this question if they reasoned that these two quantities are possibly the same because .06 is less than .6 and that $1/10$ is greater than $1/100$. The final item that was scored in a flexible fashion was the item for which students were required to shade .30 of a circle that was partitioned into 5 equal sections. Responses that .30 was close to $1/3$ and shadings of a little more than $1\ 1/2$ sections were scored as correct.

Table 1
Total Scores on the Rational Number Test, Before and After Instruction

| | Experimental <i>n</i> = 16 | | Control <i>n</i> = 13 | |
|--------------------|-------------------------------|-------|--------------------------|-------|
| | Pre | Post | Pre | Post |
| Mean score | 12.36 | 31.12 | 10.79 | 17.50 |
| Standard deviation | 6.22 | 7.46 | 5.63 | 7.67 |

Note. The maximum scores were 41 on the pretest and 45 on the posttest.

group ($t = 5.97, p < .0001$). The following two test items illustrate the difference in the explanations provided by the two groups:

- Experimenter:* Another student told me that 7 is $\frac{3}{4}$ of 10. Is it?
Experimental S1: No, because of one half of 10 is 5. One half of 5 is 2 and $\frac{1}{2}$. So if you add $2\frac{1}{2}$ to 5, that would be $7\frac{1}{2}$. So $7\frac{1}{2}$ is $\frac{3}{4}$ of 10, not 7.
Experimental S2: No, because a quarter of 10 is $2\frac{1}{2}$ and $2\frac{1}{2}$ times 3 isn't 7; it is $7\frac{1}{2}$.
Control S1: No ... 7 is not right because it is an odd number, so 6 would be right.
Control S2: Yes, 7 is $\frac{3}{4}$ of 10 because 3 plus 4 equals 7.
Experimenter: What is 65% of 160?
Experimental S1: Fifty percent [of 160] is 80. I figure 10%, which would be 16. Then I divided 16 by 2, which is 8 [5%] then 16 plus 8 um ... 24. Then I do 80 plus 24, which would be 104.
Experimental S2: Fifty percent of 160 is 80...; 25% is 40, so 75% [of 160] is 120, so it would be a little less than that [120]; it would be 10% less, so it would be about 108.
Experimental S3: Ten percent of 160 is 16; 16 times 6 equals 96. Then I did 5%, and that was 8, so ..., 96 plus 8 equals 104.
Control S1: The answer is 95 because 160 minus 65 equals 95.
Control S2: One hundred and sixty divided by 65 equals 2 remainder 30.... Is the answer 2?

A complete list of the items in this category, together with the percentage of students in each group that passed each item, is presented in Table 2.

Compare and Order

Children in the experimental group also showed significantly greater improvement than the control group on this class of item. The experimental group achieved a mean score of 5.37 (out of 7) on the posttest, as opposed to 3.45 for the control group ($t = 2.34, p < .03$). The responses to the following two items from this category illustrate the different types of understandings that the students in the two groups demonstrated and the strategies they used:

- Experimenter:* Can you think of a number that lies between decimal 3 and decimal 4?
Experimental S1: Well, point three five is between point three and point four.
Experimental S2: Decimal three zero nine.
Control S1: There is no number between decimal three and decimal four.
Control S2: Point zero three.

Table 2
Percentages of Students Succeeding on Items Requiring Some Form of Nonstandard Computation

| Items | Experimental | | Control | |
|----------------------------------|--------------|------|---------|------|
| | Pre | Post | Pre | Post |
| How much is 50% of \$8.00? | 89 | 100 | 62 | 92 |
| What is 25% of 80? | 34 | 93 | 62 | 92 |
| 15 is 75% of what? | 50 | 88 | 46 | 46 |
| Is 7 three quarters of 10? | 6 | 75 | 0 | 8 |
| How much is 10% of \$.90? | 19 | 88 | 31 | 23 |
| What is 1% of \$4.00? | 19 | 69 | 0 | 15 |
| What is 65% of 160? | 0 | 69 | 0 | 0 |
| What is .05 of 20? | 6 | 33 | 8 | 0 |
| What is 1/2 of 1/8? ^a | | 63 | | 0 |

^aNew item, administered at posttest only.

A second item from this category, “Draw a picture to show which is greater, $\frac{2}{3}$ or $\frac{3}{4}$,” also demonstrated differences between the two groups in their abilities to compare rational number quantities. The solution strategy used by student S1 in the experimental group is representative of the general approach that the students from this group used.

Experimental S1: [First the student drew two equal-sized rectangular figures.] Here is $\frac{2}{3}$ [he partitioned the first into three equal portions and shaded two parts]. Now you do the same thing for $\frac{3}{4}$ [he partitioned the second into four equal parts and shaded three parts]. So $\frac{3}{4}$ is bigger because it comes to a higher level on this one [second rectangle].

By contrast, the students in the control group had great difficulty with this item. In fact the posttest score for the control group for this item dropped slightly from the pretest score (see Table 3). The responses of the following students exemplify the ways six students in that group responded:

Control S1: They [$\frac{2}{3}$ and $\frac{3}{4}$] are both the same size because they both have one piece missing.

Control S2: Three fourths is bigger.

Experimenter: How do you know? Can you draw a picture to show me how you know?

Control S2: Here is $\frac{3}{4}$ [student drew a circle, divided it into four sections, and shaded three sections].

Here is $\frac{2}{3}$ [student drew a second circle, divided it into halves, and then divided one of the halves again, creating quarters; the student shaded the two quarters and labeled this area as $\frac{2}{3}$].

So you see that $\frac{3}{4}$ is bigger.

A final item from this category that serves as an example of the differences in the students’ thinking is one that was very difficult for both groups: *Could these be the same amount: decimal zero six of a tenth and decimal six of a hundredth?* On this item we not only asked students to demonstrate their understanding of quantity comparisons but also to work with these numbers as operators. The reasoning of the students in the experimental group demonstrates that even when

they were unable to get the correct answer, their strategies indicated that they had a considerable grasp of the underlying quantities that were involved.

Experimenter: Could these be the same amount: decimal zero six of a tenth and decimal six of a hundredth?

Experimental S1: No, they have to be different. Hey wait; no, maybe they are the same. Yes, because decimal zero six to one tenth is kinda like 6 to 100.

Experimental S2: I don't know because decimal six is bigger than decimal zero six but also one tenth is bigger than one hundredth.

By contrast, the students in the control group tended to use additive instead of ratio-based strategies, as exemplified in this response:

Control S1: No, they can't be the same because if you take 6 away [from 10], there would only be 4 left; if you take 6 away [from 100] there would be 94 left.

Table 3 shows a complete list of the items in the compare-and-order category as well as the percentage of students who succeeded on each item.

Table 3
Percentages of Students Succeeding on Items Requiring Comparison and Ordering of Numbers

| Items | Experimental | | Control | |
|--|--------------|------|---------|------|
| | Pre | Post | Pre | Post |
| Which is fewer, 1/3 or 1/2 of 6 blocks? | 44 | 100 | 69 | 92 |
| Is there a number between 0.3 and 0.4? Can you name one? | 44 | 100 | 31 | 15 |
| Which is bigger, 0.20 or 0.89? | 38 | 81 | 46 | 46 |
| Draw a picture to show which is greater, 2/3 or 3/4. | 31 | 81 | 46 | 38 |
| Place the fractions 1/2 and 1/3 on a number line. | 25 | 75 | 31 | 54 |
| Which is bigger, tenths, hundredths, or thousandths? | 6 | 56 | 23 | 85 |
| What is more, 0.06 of 1/10 or 0.6 of 1/100? | 0 | 44 | 0 | 15 |

Misleading Appearance

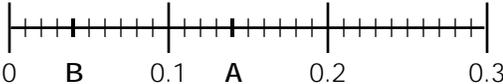
Piaget (1970) believed that unless one presented children with tasks that included misleading features, one was assessing not their true understanding but merely their abilities to parrot what they had been taught. Behr, Lesh, Post, and Silver (1983) held a similar view and suggested that misleading items are particularly revealing of conceptual understanding in the domain of rational number. We found that both groups showed some improvement on such misleading items but that the experimental group showed significantly more improvement, attaining a mean score on the posttest of 4.70 (out of 7) as opposed to 2.30 for the control group ($t = 5.67, p < .0001$). On one item students were asked to shade $\frac{3}{4}$ of a pizza that was partitioned into eight sections. The following are representative of the responses of the two groups:

Experimental S1: I don't know.... Well let me see.... This is a half [student shaded four sections],... so you would need two more to make $\frac{3}{4}$ [shades two more sections].

- Experimental S2:* There are two slices in a quarter so you need six [slices] to make three quarters [shades them].
- Experimental S3:* [Shades six sections] I just keep the quarters and forget about the eighths.
- Control S1:* It says $\frac{3}{4}$, so you need to shade in three parts [shades them].

The percentages of students correctly solving the items in this category are shown in Table 4.

Table 4
Percentages of Students Correctly Solving Items With Misleading Visual Features

| Items | Experimental | | Control | |
|--|--------------|------|---------|------|
| | Pre | Post | Pre | Post |
| Find $\frac{3}{4}$ of a pizza (predivided into 8ths). | 75 | 100 | 38 | 54 |
| Here is Mary, on her way to school. What fraction of the distance has Mary travelled from her home to school? ^a | 25 | 93 | 23 | 31 |
| What percentage of the distance has she travelled? | 25 | 82 | 8 | 38 |
| Construct the number 23.5 with base-10 blocks. ^b | 56 | 82 | 46 | 46 |
| Can you tell me what number should go at point B on the number line below? | 6 | 44 | 0 | 23 |
| Can you tell me what number should go at point A on the number line below? | 6 | 44 | 8 | 15 |
|  | | | | |
| Shade 0.3 of a circle (predivided into 5 sections). | 6 | 25 | 8 | 23 |

^aOn this problem the units into which the route is divided are small and close to each other; children tend to get very involved in counting the correct number of units she has traveled so far (5), and they neglect to count the total distance. The correct answer is $\frac{5}{8}$; the most common incorrect answer is $\frac{1}{5}$.

^bThe misleading feature here is that the long rods must be used to represent ones, and the centicube blocks to represent tenths. Students are more familiar with centicube blocks as representing the units.

Word Problems

There were only three test items in this category. As can be seen in Table 5, the students in the experimental group were more successful than the control group with these questions, achieving a mean posttest score of 2.56 (out of 3) as opposed to 1.70 for the control group. The contrast in the justifications provided by the two groups is illustrated below:

- Experimenter:* A CD is on sale. It has been marked down from \$8.00 to \$7.20. What is the discount as a percentage [of the original price]?
- Experimental S1:* I knew it was 80 cents. I did the quick math in my head and I figured out 80 cents was 10%.
- Control S1:* Eight take away seven point two zero is 80. So 80%.

The percentages of children correctly solving the three problems in this category are presented in Table 5.

Table 5
Percentages of Students Correctly Solving Word Problems

| Items | Experimental | | Control | |
|--|--------------|------|---------|------|
| | Pre | Post | Pre | Post |
| A book has 100 pages. When Jon is 90% finished reading the book, how many pages has he read? | 100 | 100 | 100 | 100 |
| Joan is 100% taller than her daughter Jessica. Jessica's height is what percentage of Joan's? (ans: 50%) | 50 | 100 | 31 | 62 |
| These CDs are marked down from \$8.00 to \$7.20. What is the discount as a percentage of the original price? | 6 | 56 | 0 | 8 |

Interchangeability of Representations

The experimental group showed significantly more improvement than the control group ($t = 3.33$, $p < .002$) on items requiring translation among fractions, decimals, and percents. The experimental group's mean score on the posttest was 5.75 (out of 9) compared to 3.07 for the control group. The strategies that students used are illustrated here:

- Experimenter:* What is $1/8$ as a decimal?
Experimental S1: Zero point one two five.
Experimenter: How did you get that?
Experimental S1: Well, $1/4$ is 25%,... and $1/8$ is half of that, so it is $12\frac{1}{2}\%$ So $12\frac{1}{2}\%$ is decimal one two and one half ... or decimal one two decimal five. No, I think it is just decimal one two five.
Control S1: I think it is decimal eight.
Experimenter: How did you get that?
Control S1: Because $1/8$ is probably the same as decimal eight.

The percentages of children successful on the problems in this category are presented in Table 6.

Table 6
Percentages of Students Succeeding on Items Requiring Movement Among Different Rational Number Representations

| Items | Experimental | | Control | |
|---|--------------|------|---------|------|
| | Pre | Post | Pre | Post |
| How much is 50% of \$8.00? | 88 | 100 | 62 | 92 |
| How many is 0.5 of all the blocks? | 56 | 100 | 77 | 77 |
| What is $1/8$ as a decimal? | 6 | 75 | 0 | 0 |
| What is $1/3$ as a percent? | 19 | 69 | 8 | 23 |
| What is seventy-five thousandths as a decimal? | 0 | 25 | 8 | 31 |
| How would you write 6% as a fraction? | 0 | 13 | 15 | 15 |
| What is 6% as a decimal? ^a | | 93 | | 23 |
| What is $1/8$ as a percent? ^a | | 75 | | 0 |
| What is thirty-five hundredths as a decimal? ^a | | 25 | | 46 |

^aNew item that was administered at posttest only.

Standard Computation

Six items were grouped together because they included a standard form of computation. On these items, the posttest performance of the two groups was similar (2.20 out of 6 for the experimental group versus 1.45 for the control group), and the statistical analysis did not reveal any significant difference. Still, because the experimental group had not been taught any formal algorithms, there were a number of interesting differences in the ways the items were approached. The following protocols illustrate the nature of these differences:

Experimenter: What is $3\frac{1}{4}$ minus $2\frac{1}{2}$?

Experimental S1: I have to carry it over, but I don't know how to carry it over, but since I'm doing a whole, shouldn't we use a quarter and a whole and then subtract a half? So the answer would be $\frac{3}{4}$.

Control S1: First I must find the common denominator, which is 4; therefore it would become $3\frac{1}{4}$ minus $2\frac{2}{4}$, which equals $1\frac{0}{4}$. So $1\frac{3}{4}$ minus $1\frac{0}{4}$; $\frac{3}{4}$.

A complete list of the standard computation items and the percentage of children who gave correct answers for each item are presented in Table 7.

Table 7
Percentages of Students Successfully Completing Items Involving Standard Computation

| Items | Experimental | | Control | |
|--|--------------|------|---------|------|
| | Pre | Post | Pre | Post |
| What is $\frac{1}{3}$ of 15? | 63 | 88 | 62 | 69 |
| How much is $0.5 + 0.38$? | 6 | 50 | 8 | 38 |
| How much is $3.64 - 0.8$? | 0 | 44 | 8 | 38 |
| How much is $\frac{2}{3}$ of $\frac{6}{7}$? | 0 | 0 | 0 | 0 |
| What is $7\frac{1}{6} - 6\frac{1}{3}$? | 0 | 0 | 0 | 0 |
| What is $3\frac{1}{4} - 2\frac{1}{2}$? | 0 | 38 | 0 | 0 |

A final illustration of the sort of thinking that children developed as a result of the experimental curriculum was provided by one student's comments while he and the interviewer worked their way through the posttest. Z had completed two pages of the six-page test, and he remarked to the interviewer, "I have just done $\frac{1}{3}$ of the test; ... that is thirty-three point three percent." When he finished the third page he noted, "Now I have finished $\frac{1}{2}$ or 50% of the test." On completing the fourth page he remarked, "Okay, so I have now done $\frac{2}{3}$ of the test, which is the same as sixty-six percent." When he had completed the penultimate page, he wondered out loud what the equivalent percentage was for $\frac{5}{6}$:

Okay, let's see; it has got to be over sixty-six decimal six percent and it is also more than 75%. I'd say that it is about 80%. . . . No wait, it can't be 80% because that is $\frac{4}{5}$ and this $[\frac{5}{6}]$ is more than $\frac{4}{5}$. It is $\frac{1}{2}$ plus $\frac{1}{3}$, . . . so it is 50% plus thirty-three point three percent, eighty-three point three percent. So I am eighty-three point three percent finished.

Supplementary Analysis

As we hope will be apparent, the qualitative differences in the thinking patterns of students in the two groups appeared on all subsets of problems, not just problems that were closest in their content to examples that had been used in the experimental program. The differences also appeared for problems on which students in both groups made errors, not just on those for which the performance of the treatment group exceeded that of the control group. This having been said, however, it seems possible that the *size* of the performance difference that emerged between the two groups on the quantitative analyses may have been a function of the facts that so many of our items involved percents and the experimental group had much greater exposure to percents than did the control group. To guard against this possibility, we performed a supplementary analysis in which we excluded all the items that had percent content.

Our new analysis was based on 28 items at pretest and 32 items at posttest (the original 28 plus 4 more). Twelve of the items on the revised scale dealt with decimals, to which both groups had considerable exposure. Sixteen pretest items and 20 posttest items dealt with fractions, to which the control group had much greater exposure. The results of this new analysis revealed that the experimental group still scored significantly higher at posttest than did the control group. On the overall measure, the experimental group achieved a mean score of 19.94 on the posttest, as opposed to 11.51 for the control group ($t = 4.07$, $p < .0003$). Because the standard deviation was approximately 5 points, these scores represent a difference in favor of the treatment group of approximately 1.5 standard deviations, which is large compared to differences typically reported in the training literature. Similar results were found when a separate analysis was conducted on a revised scale that did not include the additional 4 items added at the time of the posttest and in two further analyses in which we examined the fraction and decimal subsections of the test separately. On the fraction test, the experimental group had a mean score of 11.32 as compared to 6.92 for the control group ($t = 4.32$, $p < .0002$): a difference that is once again on the order of 1.5 standard deviations (control $SD = 2.99$). Given our program's lack of emphasis on fractions, we found the latter result both surprising and encouraging.

DISCUSSION

As long as they were asked only to perform standard procedures for manipulating very simple numbers, the children in the control group did reasonably well, both in the training and on the posttest. However, when they were confronted with genuinely novel problems, particularly ones for which a misleading cue had to be overcome or some new procedure had to be generated, they continued to make the classic mistakes that they had made on the pretest and that have been reported in the literature. Most of these mistakes involved some sort of confusion of the rational numbers with whole numbers. Thus, the responses

from the control group were symptomatic of the problem that was cited at the outset of this article: Although we educators do succeed in teaching children to manipulate rational numbers with our current instructional methods, we fail to help them develop a deep conceptual understanding of these numbers or to overcome the fundamental misconception with which they start out their learning, namely, that rational numbers are just special kinds of whole numbers.

Like the control group, the experimental group showed some improvement on the problem subset that required the application of conventional algorithms. In contrast to the control group, however, they showed a large and statistically significant improvement on the other problem sets. Not only did they get more answers correct than the control group on these other subsets, they also reasoned about the problems in ways that were qualitatively different and that demonstrated a deeper, more proportionally based understanding. Indeed, even when the experimental group students incorrectly answered a problem on the posttest, they usually made errors in which they respected the proportional status of the underlying numbers. Once again, this result was in sharp contrast to results from the control group students, who continued to treat the underlying entities as though they were whole numbers.

As was mentioned in the introduction, several other investigators have recently reported success in producing a deeper, more proportionally based understanding of fractions or decimals in the middle school years (Confrey, 1995; Kieren, 1995; Mack, 1993; Streefland, 1991). Not coincidentally, we believe, our program shares several important features with the programs designed by these other investigators. These features include (a) a greater emphasis on the meaning or semantics of the rational numbers than on procedures for manipulating them; (b) a greater emphasis on the proportional nature of rational numbers, highlighting rather than glossing over the difference between rational and whole numbers; (c) a greater emphasis on children's natural ways of viewing problems and their spontaneous solution strategies; and (d) the use of an alternative form of visual representation as a mediator between proportional quantities and their conventional numeric representations (i.e., an alternative to the standard pie chart).

Although the present program was similar in many respects to other recent experimental programs, it also had a number of distinctive features. First and foremost, we believe, was the fact that it attempted to move children beyond the understanding of any single form of rational number representation toward a deeper understanding of the rational number system as a whole. The effect of this latter emphasis was most clearly demonstrated on the problem subset that focused on the interchangeability of representations. As might be expected, these problems were ones in which the difference between the performance of the experimental and the control groups was particularly large. In this subset, the only items that posed a difficulty for experimental children were those that required conversion of a fraction such as $1/100$ or $1/1000$. This latter difficulty was not one that we had anticipated. However, we attribute the problem to the fact that we ran out of time at the end of the program and were not able to work

longer on fractions. In subsequent replications we have been more successful in helping the students to work with items such as these (Moss, 1997).

Although children's understanding of the overall system was most apparent on the problems requiring direct conversion from one form of representation to another, their understanding was also apparent in most of the other problem subsets. For example, in compare-and-order problems, when the children were asked if there is a number between 0.3 and 0.4, the experimental children treated the answer to the question as obvious and saw the only difficulty as one of actually generating such a number (a difficulty they coped with in various ingenious ways). Similarly, regardless of whether they were dealing with fractions or decimals or both, the experimental children treated the request to order several numbers as one that required them to think in terms of the underlying ratios that were involved rather than in terms of their whole number knowledge. Again, they used various methods to represent these entities, but they never used the sort of simple whole number strategy that was characteristic of the control group. Finally, in the problems that involved nonstandard computation, the experimental children again used a wide variety of strategies, not just the simple halving strategy that they had been taught. Their responses thus indicated that they had acquired the sort of confidence, flexibility, and inventiveness that have been called for in the literature under the rubric of number sense (National Council of Teachers of Mathematics, 1989; Sowder, 1995).

To the best of our knowledge, the only other program that has attempted to produce this sort of generalized understanding of and flexibility in working with the rational numbers is the very successful longitudinal study conducted by Confrey and her colleagues (Lachance & Confrey, 1995; Scarano, 1996). Like most reform approaches, Confrey's program was focused heavily on meaning and on children's spontaneous efforts to construct it. In keeping with others' reform approaches, its developers emphasized that rational numbers must be treated as proportions and used other forms of representations beyond simple pie charts. Finally, as in the present approach (in contrast to other approaches), a strong emphasis was placed (a) on continuous quantity and measurement, as opposed to discrete quantity and counting; (b) on splitting as a natural form of computation that can be used in a measurement context; and (c) on the equivalence among different forms of rational number representation. Given the convergence of the two programs on these points, they would seem likely candidates for inclusion in future efforts to reform the teaching of rational number on a more widespread basis.

In what ways did the two programs differ? The most obvious difference is that Confrey's program spanned 3 years, whereas our program involved a total of 20 lessons spread across a 5-month period. We would not want to leave the impression that we think a short intervention of the sort that we attempted is preferable to a long intervention of the sort mounted by Confrey and her colleagues. To the contrary, given the importance of the rational numbers in higher mathematics and the depth of children's misconceptions, we think that a longer and more sus-

tained intervention—one that pervades the entire middle school curriculum—would be desirable.

Still, given the short nature of our intervention and the powerful nature of the change that it stimulated, it seems worthwhile to ask what other features of our program were distinctive and might have led to the rapid “uptake.” Our conjecture is that the following three features were particularly effective:

1. We began our program with percents, thus permitting children to take advantage of and to combine their qualitative understanding of proportions and their knowledge of the numbers from 1 to 100 although avoiding (or at least postponing) the problems that fractions present, either on their own or as a basis for understanding decimals; in effect our program mobilized children’s whole number knowledge in such a fashion that it aided their understanding of rational numbers, rather than interfering with it.
2. In our program we used a unidimensional form of number representation (the number ribbon) in a context that emphasized the global, proportional nature of any quantity, rather than the multiple units or “shares” into which it may be divided.
3. In our program we emphasized benchmark values for moving among the different forms of representation, an emphasis that permitted students to think about problems in a much more flexible fashion and to use procedures of their own invention for approaching the problems.

As evidence for the importance of the first feature (beginning with percents), we would point out that, in the protocols that Confrey and her colleagues have presented, children often use the language of percents, even though that language was not emphasized in the program to which they were exposed (Lachance & Confrey, 1996). Our conjecture is that percents may have the same status with regard to the representation of proportions that splitting has with regard to operations on proportions: They constitute a way of thinking about such numbers that is natural and powerful. Not only do percents constitute an excellent entry point into this difficult domain, they provide an excellent anchor point for students while they proceed through the domain, one that we think should be emphasized.

In many respects the present results fit well with those reported elsewhere. In keeping with the work of other investigators and on the basis of our results, we suggest that in any successful attempt to teach a deeper understanding of rational numbers, educators must place a stronger emphasis on the semantics of the rational numbers and on their proportional nature, in a context that is child centered and that uses some alternative form of representation in addition to, or instead of, the standard pie diagram. Like the results reported by Confrey and her colleagues, our results provide evidence for the desirability of focusing on continuous quantity and measurement rather than on discontinuous quantity and counting, of emphasizing the relationships among different ways of representing proportional quantities (fractions, decimals, percents), and of providing practice in apply-

ing the same intuitive forms of operating (especially halving and “10-splitting”) across these different representational forms.

The way in which this study departs most sharply from previous studies, including Confrey’s, is in showing that children can be helped to construct a rapid and serviceable overview of the rational numbers from the time of their first introduction to them. The approach that we developed for doing so is to begin with percents and decimals rather than fractions, use a simple unidimensional representation of number, and focus on benchmark equivalencies among percents, decimals, and fractions throughout the curriculum. Our conjecture is that if these techniques were adapted on a more widespread basis, educators would be better able to capitalize on children’s accomplishments by the time they reach middle school, on the one hand, and to lay a more solid foundation for the future, on the other.

REFERENCES

- Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational-number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91–126). New York: Academic Press.
- Carpenter, T. P., Corbitt, M. K., Kepner, H. S., Jr., Lindquist, M. M., & Reys, R. E. (1980). National assessment: A perspective of students’ mastery of basic mathematics skills. In M. M. Lindquist (Ed.), *Selected issues in mathematics education* (pp. 215–227). Chicago: National Society for the Study of Education and Reston, VA: National Council of Teachers of Mathematics.
- Case, R. (1985). *Intellectual development: Birth to adulthood*. Orlando, FL: Academic Press.
- Confrey, J. (1994). Splitting, similarity, and the rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 291–330). Albany: State University of New York Press.
- Confrey, J. (1995). Student voice in examining “splitting” as an approach to ratio, proportions and fractions. In L. Meira & D. Carraher (Eds.), *Proceedings of the 19th international conference for the Psychology of Mathematics Education* (Vol. 1, pp. 3–29). Recife, Brazil: Universidade Federal de Pernambuco.
- Griffin, S., & Case, R. (1996). Evaluating the breadth and depth of training effects when central conceptual structures are taught. In R. Case & Y. Okamoto (Eds.), *The role of central conceptual structures in the development of children’s thought. Monographs of the Society for Research in Child Development, 61*, (1–2, Serial No. 246, pp. 83–102). Chicago: University of Chicago Press.
- Hiebert, J. (1992). Mathematical, cognitive, and instructional analyses of decimal fractions. In G. Leinhardt, R. Putnam, & R. A. Hattrop (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 283–322). Hillsdale, NJ: Erlbaum.
- Hiebert, J., & Wearne, D. (1986). Procedures over concepts: The acquisition of decimal number knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 199–223). Hillsdale, NJ: Erlbaum.
- Hiebert, J., Wearne D., & Taber, S. (1991). Fourth graders’ gradual construction of decimal fractions during instruction using different physical representations. *Elementary School Journal, 91*, 321–341.
- Kerslake, D. (1986). *Fractions: Children’s strategies and errors*. Windsor, England: NFER-Nelson.
- Kieren, T. E. (1992). Rational and fractional numbers as mathematical and personal knowledge. In G. Leinhardt, R. Putnam, & R. A. Hattrop (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 323–371). Hillsdale, NJ: Erlbaum.
- Kieren, T. E. (1994a). Multiple views of multiplicative structure. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 387–397). Albany: State University of New York Press.

- Kieren, T. E. (1994b). Reflections and interactions on rational number thinking, learning and teaching. In D. Kirshner (Ed.), *Proceedings of the sixteenth annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 53–56). Baton Rouge: Louisiana State University.
- Kieren, T. E. (1995). Creating spaces for learning fractions. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 31–65). Albany: State University of New York Press.
- Lachance, A., & Confrey, J. (1995). Introducing fifth graders to decimal notation through ratio and proportion. In D. T. Owens, M. K. Reed, & G. M. Millsaps (Eds.), *Proceedings of the seventeenth annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 395–400). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.
- Lachance, A., & Confrey, J. (1996, April). *Mapping the journey of students' explorations of decimal notation through ratio and proportion*. Paper presented at the annual meeting of the American Educational Research Association, New York.
- Lembke, L. O., & Reys, B. J. (1994). The development of, and interaction between, intuitive and school-taught ideas about percent. *Journal for Research in Mathematics Education*, 25, 237–259.
- Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education*, 21, 16–32.
- Mack, N. K. (1993). Learning rational numbers with understanding: The case of informal knowledge. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 85–105). Hillsdale, NJ: Erlbaum.
- Markovits, Z., & Sowder, J. T. (1991). Students' understanding of the relationship between fractions and decimals. *Focus on Learning Problems in Mathematics*, 13 (1), 3–11.
- Moss, J. (1997). *Developing children's rational number sense: A new approach and an experimental program*. Unpublished master's thesis, University of Toronto, Toronto, ON, Canada.
- National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.
- Noelting, G. (1980a). The development of proportional reasoning and the ratio concept. Part I: Differentiation of stages. *Educational Studies in Mathematics*, 11, 217–253.
- Noelting, G. (1980b). The development of proportional reasoning and the ratio concept. Part II: Problem-structure at successive stages: Problem-solving strategies and the mechanism of adaptive restructuring. *Educational Studies in Mathematics*, 11, 331–363.
- Nunes, T., & Bryant, P. (1996). *Children doing mathematics*. Cambridge, MA: Blackwell.
- Ohlsson, S. (1988). Mathematics meaning and applicational meaning in the semantics of fractions and related concepts. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (Vol. 2, pp. 53–92). Hillsdale, NJ: Erlbaum and Reston, VA: National Council of Teachers of Mathematics.
- Okamoto, Y., & Case, R. (1996). Exploring the microstructure of children's central conceptual structures in the domain of number. In R. Case & Y. Okamoto (Eds.), *The role of central conceptual structures in the development of children's thought. Monographs of the Society for Research in Child Development*, 61, (1–2, Serial No. 246, pp. 27–58). Chicago: University of Chicago Press.
- Parker, M., & Leinhardt, G. (1995). Percent: A privileged proportion. *Review of Educational Research*, 65, 421–481.
- Piaget, J. (1970). Piaget's theory. In P. H. Mussen (Ed.), *Carmichael's manual of child psychology* (3rd ed., Vol. 1, pp. 703–732). New York: Wiley.
- Post, T. R., Cramer, K. A., Behr, M., Lesh, R., & Harel, G. (1993). Curriculum implications of research on the learning, teaching and assessing of rational number concepts. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 327–362). Hillsdale, NJ: Erlbaum.
- Resnick, L. B. (1982). Syntax and semantics in learning to subtract. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 136–155). Hillsdale, NJ: Erlbaum.

- Resnick, L. B., & Singer, J. A. (1993). Protoquantitative origins of ratio reasoning. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 107–130). Hillsdale, NJ: Erlbaum.
- Scarano, G. H. (1996, April). *Results from a 3-year longitudinal teaching experiment designed to investigate splitting, ratio and proportion*. Paper presented at the annual meeting of the American Educational Research Association, New York.
- Sowder, J. T. (1995). Instructing for rational number sense. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 15–30). Albany: State University of New York Press.
- Streefland, L. (1991). Fractions: An integrated perspective. In L. Streefland (Ed.), *Realistic mathematics education in primary school* (pp. 93–118). Utrecht, The Netherlands: Freudenthal Institute.
- Streefland, L. (1993). Fractions: A realistic approach. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 289–325). Hillsdale, NJ: Erlbaum.

Authors

Joan Moss, Doctoral candidate, Department of Human Development and Applied Psychology, Ontario Institute for Studies in Education, University of Toronto, 45 Walmer Road, Toronto ON M5R 2X2, Canada; jmoss@oise.utoronto.ca

Robbie Case, Professor, Institute of Child Study, University of Toronto, 45 Walmer Road, Toronto ON M5R 2X2, Canada; RobbieCase@aol.com