Introducing Percents in Linear Measurement to Foster an Understanding of Rational-Number Operations

Joan Moss

How do we foster computational fluency with rational numbers when this topic is known to pose so many conceptual challenges for young students? How can we help students understand the operations of rational numbers when their grasp of the quantities involved in the rational-number system is often very limited? Traditional instruction in rational numbers focuses on rules and memorization. Teachers often give students instructions such as, "To add fractions, first find a common denominator, then add only the numerators" or "To add and subtract decimal numbers, line up the decimals, then do your calculations."

Remembering and properly executing these relatively simple rules can be very difficult, however, when students do not have a good conceptual grounding in the rational numbers. For example, a recent study found that sixty-five percent of students in a random sample of twenty sixth-grade students had difficulty with addition of fractions. Furthermore, only ten percent of the same group could explain how fraction addition worked, even though they all had learned the rules over a period of several years (Kilpatrick, Swafford, and Findell 2001). Students often erroneously interpret a fraction as two independent and countable numbers. When asked to find the answer to a problem such as 1/2 + 1/3 = ____, they assert that the answer is 2/5—a number that is less than the addend 1/2.

Many students also confuse the rules related to decimal operations. Heibert and Wearne (1986) reported that when asked to find the sum of 4 + .3, a majority of middle school students were unable to find a correct answer, and a majority of these students asserted that the answer is .7. Clearly, unless students achieve the necessary understanding of the quantities involved in rational numbers and the meaning of the symbols and operations, they may experience significant difficulties with rational-number computation.

This article presents computation abilities in rational numbers that were developed by fourth-grade students who participated in a research project on the learning of rational numbers (Moss and Case 1999). The goal of this ongoing research project is to foster a flexible, interconnected understanding of the rational-number system (see also Kalchman, Moss, and Case 1999; Moss 2002; Moss 2000). The project does not involve teaching students any particular rules for operations with rational numbers; instead, its focus is to help students gain a "number sense" approach to rational number (Sowder 1995) so that they are able to move among fractions, decimals, percents, and ratios in a seamless or flexible manner and to invent procedures for calculating with these numbers.

The project's instructional approach to teaching this number system is unusual. Students begin their investigations of rational

Joan Moss teaches in the Department of Human Development and Applied Psychology at the Ontario Institute for Studies in Education, University of Toronto. Her research is on the development of number sense and rational-number understanding, as well as the role of metacognition in mathematics learning.

Edited by **Carol Langbort,** *clangbo@sfsu.edu, Department of Elementary Education, San Francisco State University, San Francisco, CA 94132. Readers are encouraged to send manuscripts appropriate for this section to "Research, Reflection, Practice," NCTM, 1906 Association Dr., Reston, VA 20191-1502.*

numbers by learning about percents in the context of linear measurement. The learning of fractions and decimals comes later and is grounded in students' learning of percents. Although this approach alters the traditional teaching sequence for rational numbers, it has many advantages for young students. For example, by beginning with percents rather than fractions, educators postpone the problem of students' having to compare or manipulate ratios with different denominators. This allows children to concentrate on developing their own procedures for comparison and calculation, instead of struggling to master a complex set of procedures that might seem foreign to them.

Although every percentage value has a corresponding fractional or decimal equivalent that is easy to determine, the converse is not true. Fractions such as 1/7 and 1/9 do not have an easily calculated decimal or percent equivalent. Therefore, a second advantage of this instructional approach is that by beginning with percents, we allow children to make their first conversions among the different representations of rational numbers in a direct and intuitive fashion and to develop a better general understanding of how the three systems are related. Finally, students not only appear to have substantial informal knowledge of percents in daily life but also show a good intuitive sense for operating with these numbers (Lembke and Reys 1995).

This article describes the kinds of activities that the students completed to develop their conceptual understanding of and computational abilities for the rational-number system. Each time this curriculum was implemented, we varied some of the lessons to accommodate the children's levels of understanding. The sequence of instruction remained the same, however, as did the specific props and representations that we used. An excerpt from one of the lessons that took place toward the end of the twenty-session learning cycle will illustrate the kinds of strategies that the students developed for calculating with rational numbers.

At the time of the lesson, the students had been working with their teacher on strategies for combining and subtracting rational numbers of mixed representations. Pairs of students were invited to come to the front of the class to present addition and subtraction challenges that they had designed. The students worked with these calculation challenges orally, mostly without using pencil and paper. The students had developed several nonstandard methods for working with these numbers. Two students, Janet and John (pseudonyms), were the first to present their challenges. John wrote the challenge on the board: 1/4 + .125 + 1/8 + .5 = _______.

Sam. OK, I can do that. It's easy. The answer is 1, 'cause point one two five and 1/8 is .25, and another quarter [0.25] makes .5. Then you still have another half, so that makes one whole. So the answer is 1.

The next child to speak, Iris, also arrived at the same answer but translated the mixed representations in the number sentence into their fraction form rather than into decimals.

Iris. I also think the answer is 1, but I did it like this: First of all, I changed 1/2 to 2 quarters, so now you start with 2 quarters. Then you add the first 1/4, and now you have 3 quarters. Also, if you add .125 and 1/8, that makes another quarter. So 4 quarters makes 1.

The challenge that Simon and Louisa posed was what we called a "true or false" question.

Simon. So here is ours. Do you think this is true or false? [Louisa proceeded to write the following: 37 1/2% − 1/8 − 0.10 = 0.15.]

Jessica. Let's see. It's a bit hard. An eighth is 12 1/2 percent, so take the 12 1/2 percent from 37 1/2 percent and that's 25 [percent]. Now take away 10 percent and the answer is 15 percent. So it's true; it's point 15.

Finally, Ryan and Sascha came to the front of the class. They had chosen to use the question format of "How many more to make one whole?" This format was a favorite with the students, because their tendency was to make very long and intricate strings of numbers. The format also allows students to answer in many different ways.

Sascha. So we want to know what you need to add to the numbers to make a whole: 1/16 + 1/16 + .25 + 15% + 1/4 + 1/8 + _______ = 1.

Ellen. One-sixteenth and 1/16 is 12 1/2, and then with 1/8, that is 25. So add another 25, that is 50; then 1/4, that is 75. Now add the 15 [percent] and that is 90. So the whole thing adds to 90. So you need 10 percent or point 10 more to get one whole.

Ellen had a tendency to omit the words "percent" and "decimal" as she worked through her solution. Although this practice, which was typical of many other students, was not encouraged, it did not appear to deter the students from understanding the quantities involved or producing correct answers.

These examples are drawn only from a single research classroom, but they share features of reasoning that developed in other classrooms in which this intervention has been implemented (see Kalchman, Moss, and Case 2001). Most significant among these features are the following:

- the apparent ease with which the students move among the representations of rational number;
- the use of benchmarks rather than standard procedures to translate among the representations of fractions, decimals, and percents; and
- the strong sense of the quantities involved in the rational number representations.

How did this kind of reasoning develop? The next section of this article will briefly outline the curriculum sequence to show how our earliest activities with percents and measurement led to the way that students worked with rational numbers.

Percents and Beakers

Instruction began with students first estimating, then calculating, the percentage of water contained at various levels in assorted beakers.

Teacher. I have filled this beaker with blue water. Can you estimate what percent full this beaker is?

Student. I guess that the beaker is approximately 25 percent full.

Teacher. Then how would you find out exactly how to measure the water to the 25-percent line?

Student. Easy. The beaker is 12 centimeters tall, so if it was 50 percent full, the line would be halfway up the container, and that would be 6 centimeters. But it is 25 percent full, so you have to do half again. So that would be 3 centimeters.

The students naturally chose this repeated halving strategy to solve the early measurement challenges. When the measurement exercises were expanded to include quantities that represent 75 percent, the students once again developed their own methods to find a solution. They divided the task into a series of steps. Jessica's solution for calculating 75% × 18 typifies the steps that most of the students went through.

Jessica. Well, to get 75 percent of 18, first you have to get 50 percent, which is 9; and then half of that, 25 percent, is 4 1/2. So you add the two together and you get 13 1/2. So 75 percent of 18 is 13 1/2.

The benchmarks of 25 percent, 50 percent, and 75 percent became standard tools in all the students' subsequent work, as did the names of the same quantities: one-quarter, one-half, and three-quarters. The students were shown how to write these fractions in symbolic form: 1/4, 1/2, and 3/4.

A New Benchmark

As students continued to measure and calculate the fullness of various quantities or percents of lengths of other objects in the classroom, they naturally included the new benchmark of 12 1/2 percent (half of 25 percent). This new benchmark, always associated with its fractional equivalent of 1/8, became useful as a tool for calculating quantities. In the same way that they had performed a sequence of halving and combining to calculate 75-percent quantities, the students now could broaden their calculations to include new quantities such as 37 1/2 percent (25% + 12 1/2%) and 62 1/2 percent (50% + 12 1/2%).

Decimals and Stopwatches

Once the students had reached a level of proficiency with the percent benchmarks, we introduced them to two-place decimals in their relation to percents. The basic approach was to show that a two-place decimal number represents a percentage of the "way" between two adjacent whole numbers; for example, 5.25 is located 25 percent of the way between 5 and 6. We used digital stopwatches that display seconds and hundredths of seconds; the latter are indicated by two small digits to the right of the numbers. We considered what the two "small numbers" might mean and how these small numbers are related to the greater numbers to the left (the seconds). After experimenting with the stopwatches and noting that one second contained one hundred of these small units of time, the students made the connection to percents with comments such as, "It's like they are percents of a second."

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The stopwatches with their hundredths of seconds served as a powerful concrete representation of decimals and an excellent tool for many activities for comparing and computing with rational numbers. In an opening activity, the "Stop/Start Challenge," students attempted to start and stop the watch as quickly as possible, several times in succession. They were taught to record their times as decimals; for example, twenty hundredths of a second was written as .20, nine hundredths of a second as .09, and so on. To compare their personal fastest reaction time with that of their classmates, students had the opportunity not only to order decimal numbers but also to find ways to compute differences in decimal numbers, or scores. Another stopwatch game that fostered an understanding of magnitude was "Stop the Watch Between." The game included questions such as, "Can you stop the watch between .45 and .50?"

Finally, students played games in which the object was to move between representations of rational numbers and calculate sums and differences. In a game that we designed called "Crack the Code," students were asked to stop the watch at the decimal equivalent of different fractions and percents. For example, if given the secret code "3/4," students were required to stop the watch at seventy-five hundredths of a second. In other secret-code challenges, the students performed computations. They were asked to stop the watch as close as possible to the answer to an equation—for example, 1/2 + 3/4 = 1.25—and then "figure out the decimal value for how close you were."

The stopwatch activities and many others were not only useful in promoting children's sense of equivalencies and mixed representations but also enjoyable. In the final days of the intervention, the students devised stopwatch activities of their own. Moss (2000) includes a complete list of both the content covered in each lesson and the challenge problems that were assigned.

Assessing Performance of Standard Computation

When we assessed the students' performance on rational-number tasks following the interventions, we found that the majority of the students outperformed students of their own age as well as older students on tasks that required conceptual understanding. Although we were encouraged by this outcome, we had anticipated it. The students' performance on traditional types of computation tasks was more surprising; we found that the experimental students performed as well as or better than students whose instruction had been based on learning rules for computation.

What follows are illustrations of students' work with standard computation. Below are two examples of students' reasoning on a subtraction item from one of the post-test measures.

Interviewer. What is 3 1/4 − 2 1/2?

Experimental Student 1. I am not sure. I have to carry it over, but I don't know how to carry it over. But since I'm doing a whole, shouldn't we use a quarter and a whole and then subtract a half? So the answer would be 3/4.

Experimental Student 2. Um, 3 and a quarter is 3.25, and 2 and a half is 2 point fifty. So you just do a subtraction: 3.25 − 2.50 = .75.

By contrast, a student from the control classroom tried to use remembered rules and produced an answer that did not make sense. One student's answer showed the kinds of reasoning that these students produced: "First, I must find the common denominator, which is 4; so it is 3 1/4 – 2 2/4, so the answer is 1 and 0/4."

The following example of student reasoning on a fraction multiplication item is taken from a post-test interview of a high-achieving student and is not representative of all the students' thinking by the time of the post-test. I include this example, however, to show how our approach to rational-number teaching, with its emphasis on the meaning of rational-number operations, can prompt students to reason about operations that are difficult even for most adults (Moss 2000).

Interviewer. How much is 2/3 of 6/8?

Student. Well, 6/8 is really three-quarters, so that is 75 percent. Well, one-third of 75 percent is 25 percent. But you need 2/3. So it is 50 percent, and that's one-half.

Closing Thoughts

The definition of computational fluency in NCTM's *Principles and Standards* (2000) includes three ideas: efficiency, accuracy, and flexibility. Embedded in these ideas is the requirement that students have knowledge of basic number combinations and important number relationships and the ability to choose from among a number of appropriate strategies. Furthermore, to be computationally fluent students should demonstrate an understanding of the meanings of the operations and the relationships of these operations to one another (Russell 2000).

The students in our research achieved a kind of computational fluency and revealed that they have developed many of these requirements. As this article has illustrated, the way that the students use their knowledge of benchmarks and repeated halving appears to give them a strong sense of the magnitude and relationships of rational-number quantities as well as an understanding of the meaning of the operations. Although these strategies eventually lead the students to find correct answers, it is also true, however, that the methods that the students use are not as efficient as more standard algorithms. In our upcoming research, we plan to extend this intervention to include more standard procedures. This will help us discover whether the kinds of underlying understanding of the operations that the students have gained and the confidence that they show in working with rational numbers will ultimately lead them to perform standard computation with understanding when they move to working with numbers in a more abstract realm.

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