Prisms and Pyramids:

Constructing Three-Dimensional Models to Build Understanding

Beverly A. Koester

he Geometry Standard of NCTM's *Principles and Standards for School Mathematics* (2000) describes the study of both two- and three-dimensional shapes. In the elementary grades, plane geometry often receives more emphasis than does three-dimensional geometry. In many classrooms, students actively explore two-dimensional shapes using a rich variety of materials, including tangrams, geoboards, and pattern blocks. Materials and lessons for exploring three-dimensional shapes are not as prevalent.

This article describes how students in grades three to five can actively explore three-dimensional shapes using simple, inexpensive materials: paper and drinking straws. The article first discusses basic principles that underlie geometry instruction, then presents a variety of experiences that focus on the properties of prisms and pyramids and the relationships among different shapes.

The Van Hiele Model of Geometric Thinking

The van Hiele model of geometric thinking guided me in making instructional choices for my fourth-grade students (van Hiele 1999). Based on their experiences as teachers in the Dutch public schools, Pierre and Dina van Hiele developed the theory that students progress through the following distinct levels of development in their geometric thinking:

Visual level. Students judge figures by their appearances. A child might say, "It is a rectangle because it looks like a box."

Descriptive level. Students analyze figures based on properties and attributes. They are able to make generalizations about specific shapes, but they cannot make generalizations about how different properties of a shape relate to one another or how different types of shapes relate to one another.

Informal-deduction level. Students can generalize interrelationships of properties within the same shape and among different shapes. They can understand and develop abstract definitions, and they can meaningfully classify shapes in hierarchies.

The van Hiele theory asserts that the development of geometric thinking is more dependent on instruction than on age. The van Hieles propose

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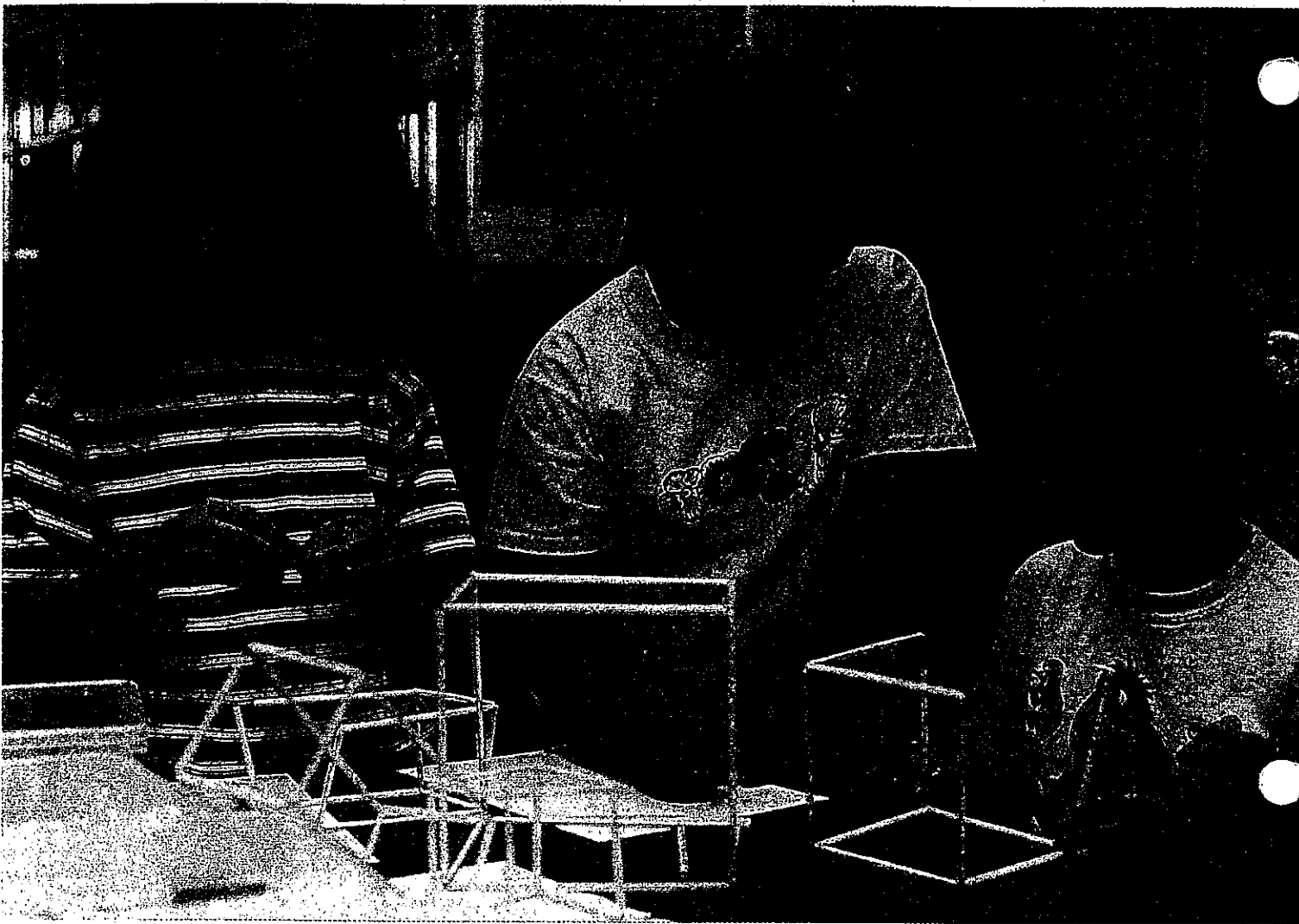
sequential phases of learning to help students move from one level to another. Instruction should include a sequence of activities that begins with "play" and exploration, gradually builds concepts and related language, and culminates in summary activities that help students integrate new ideas into their previous knowledge.
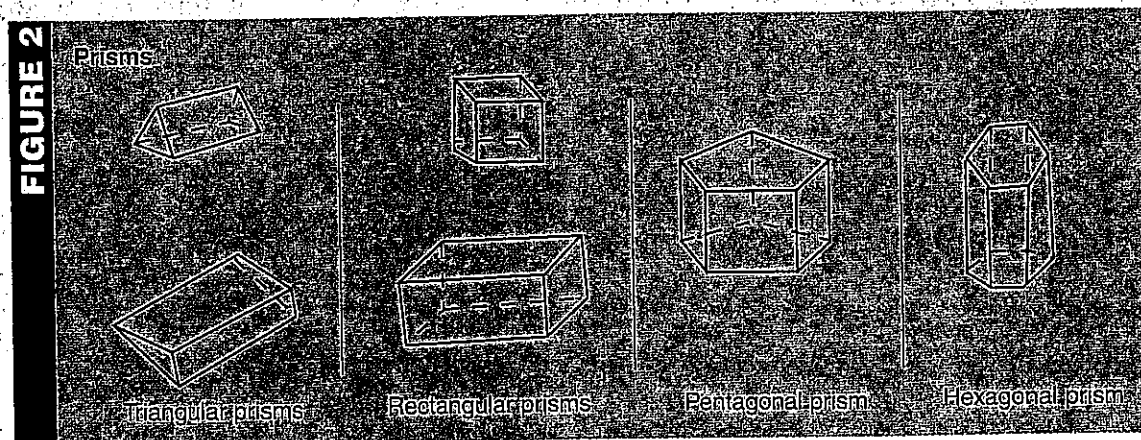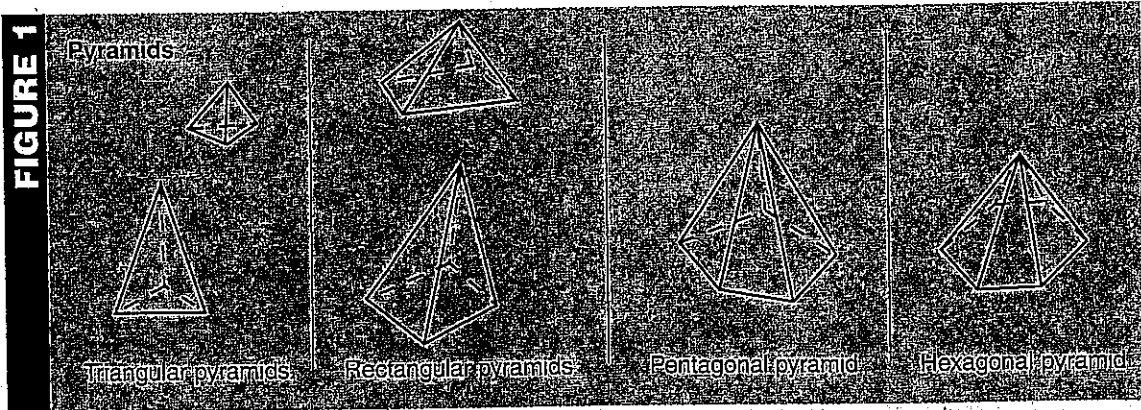
Background of My Students

Prior to studying three-dimensional geometry, my students had explored topics in plane geometry through a variety of hands-on activities. I recognized that most students could analyze component parts of two-dimensional figures, consistent with the van Hiele descriptive level of thinking. Some students were moving toward the informal-deduction level of thinking, in which students logically order properties.

Through informal assessment, I found that many students were still thinking about three-dimensional shapes at the visual level. This did not surprise me

because students often think about different concepts at different levels, depending on their experience. Although many students were familiar with mathematical names for three-dimensional figures (such as *cube, sphere, cylinder,* and *cone*), they had not developed the language to describe properties of these shapes (such as *shape of faces* and *number of edges*). Students did not have enough experience to make generalizations about classes of shapes. For example, I explained to one student that a cube is a special kind of prism. She responded, "I know this is a cube. If it is a cube, how can it also be a prism? I thought a prism has to have a triangle at the ends." Some students knew the terms *prism* and *pyramid,* but they associated these terms with particular instances of the shapes rather than with an entire family of shapes (see **figs. 1** and **2**).

NCTM Standards for Geometry

I wanted to help my students move from a visual level to higher levels of thinking about three-

FIGURE 1

Pyramids

Triangular pyramids   Rectangular pyramids   Pentagonal pyramid   Hexagonal pyramid

FIGURE 2

Prisms

Triangular prisms   Rectangular prisms   Pentagonal prism   Hexagonal prism

dimensional shapes. I knew that students would need experience with many different examples of prisms and pyramids in order to generalize the common features of these classes of shapes. I wanted to plan activities that would engage students in building and drawing, as well as in thinking and talking about geometric properties and relationships.

Consistent with the van Hiele theory, the NCTM Standards advocate an active approach to learning, with access to a variety of concrete materials. NCTM's Geometry Standard for grades three to five describes specific expectations for student learning (NCTM 2000). In developing activities for my students, I focused primarily on the following expectations:

In grades three through five, all students should—

- identify, compare, and analyze attributes of two- and three-dimensional shapes and develop vocabulary to describe the attributes (descriptive level);
- classify two- and three-dimensional shapes according to their properties and develop definitions of classes of shapes such as triangles and pyramids (informal deduction level);
- explore congruence and similarity; and
- build and draw geometric objects.

Our First Model: Folding Paper to Form Faces of Three-Dimensional Shapes

Rectangles to cylinders and prisms

What happens when you roll up a rectangular piece of paper? You form the curved surface of a cylinder. Now suppose you first fold the rectangle along a series of parallel lines, and then overlap one or more sections. In this way, you can form the lateral faces of a prism (see **fig. 3**). Although this model does not include the two bases of the prism, visualizing these missing surfaces is easy.

I organized my students into groups of four and gave each group a set of paper rectangles of different sizes, scissors, and tape. I asked each group to create at least one model of a triangular prism, rectangular prism, pentagonal prism, and hexagonal prism. We discussed how a prism is named for the shape of its base.

Circles to cones and pyramids

Just as cylinders can be related to prisms, so can cones be related to pyramids. The students cut out a circle, cut along one radius, then formed a cone. Then they cut out regular octagons. By overlapping different numbers of sections, the students created

regular pyramids with different bases: triangular, rectangular, pentagonal, hexagonal, and heptagonal (see fig. 4). Some students continued this exploration by starting with squares and hexagons instead of octagons.

Describing Prisms and Pyramids: A Beginning

After students had created paper models, I posted two large sheets of paper with the headings "Properties of Prisms" and "Properties of Pyramids." My goal was for students to begin to observe and compare common characteristics of all prisms and all pyramids, based on the models that they had created. As students made observations, I introduced terminology—*face, edge, vertex,* and *plane*—to help them describe characteristics with more precision. We also reviewed terminology that students had learned in their study of two-dimensional shapes, such as *polygon, parallel,* and *perpendicular.* A typical conversation follows.

Student 1. A prism is like a cylinder, except that the top and bottom are different shapes instead of circles.

Teacher. The end faces of a cylinder or a prism are called *bases.* What can you say about the bases of all the prisms you have made? What do these shapes have in common?

Student 1. They have straight sides. They are all polygons.

*Student 2.* The two bases are the same size and shape.

Teacher. So we could say that the bases of a prism are *congruent* polygons.

Student 3. Here's another thing: A cylinder is curved between the bases. A prism has rectangles that connect the bases.

Teacher. Is this true for all prisms? Let's look at the paper models. Are these lateral faces [pointing] all rectangles? [The students respond, "Yes."] Tomorrow we are going to create new models using straws and connectors. You may find out that building a prism with lateral faces that are not necessarily rectangles is possible.

Our Second Model: Connecting Straws to Form Edges of Three-Dimensional Shapes

The next day, I showed my students a model of a rectangular prism that I had constructed by attaching plastic drinking straws with twist-tie connectors. I asked students to compare this model with a

FIGURE 3
Rectangles to cylinders and prisms

FIGURE 4
Circles to cones and pyramids
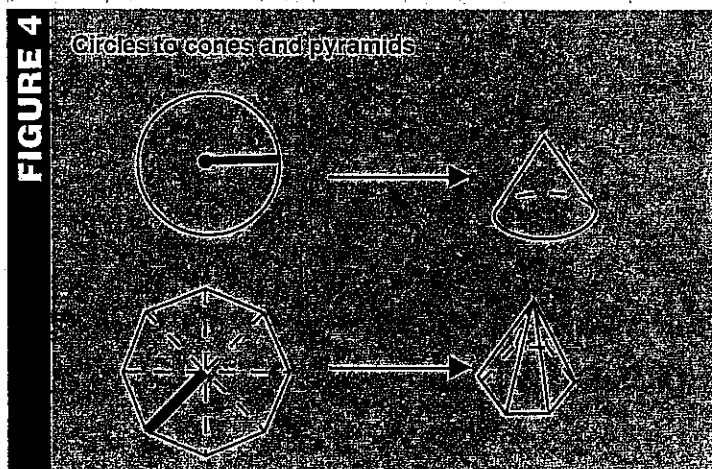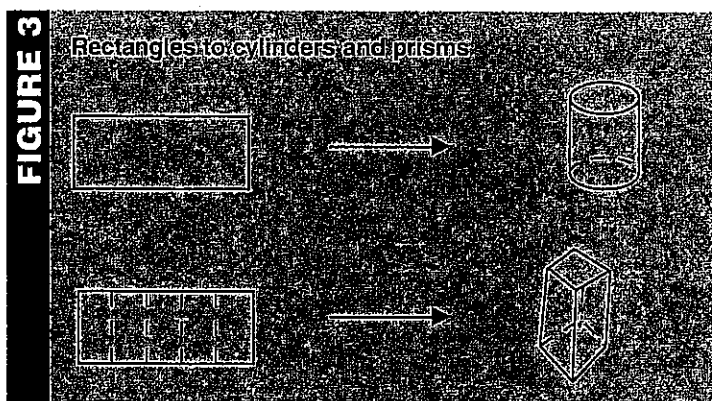
paper model of a prism. The paper model shows the prism's *faces.* The straw model shows the prism's *edges.* By definition, a prism consists of all the edges as well as all the points within the interior of each face that the edges form.

Materials and preparation

We used one thousand small-diameter plastic drinking straws, an ample quantity for whole-class investigations. I involved my students in measuring and cutting the straws in three different lengths—three inches, five inches, and seven inches. Using straws of different lengths enables students to create a greater variety of shapes than does using straws of a uniform length. For connectors, we used a box of two thousand four-inch twist-ties that are often used to fasten plastic bags. I was able to buy the twist-ties at a local grocery store for a low price. Pipe cleaners also work well as connectors, but they are more expensive. Both the straws and the connectors are reusable. My students understood that they would be able to make several three-dimensional shapes and that they then would need to dismantle them in order to explore new problems.

Exploration

My students were eager to build. I first gave them

time to explore the new materials freely, keeping in mind Pierre van Hiele's statement that "geometry begins with play." Students built models of real structures, including the Great Pyramid at Giza, playground equipment, and houses with different types of roofs. As they worked, I listened to various questions and observations, such as "How many straws will I need? Do they all have to be the same length? Look: At the top of the pyramid I have to connect four different straws at one point."

FIGURE 5

Connecting straws
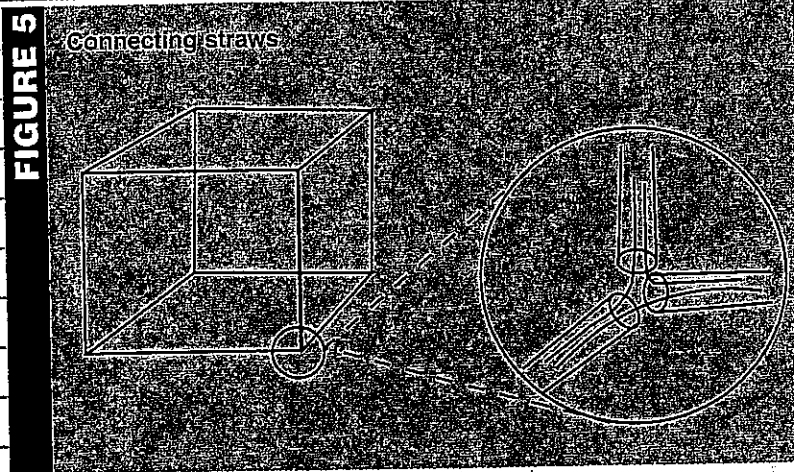
Most students had little trouble figuring out how to connect three or more straws at one point. Figure 5 shows one technique. If you use straws with a greater diameter, you can fold back about an inch of each connector for a tighter joint.

Oblique prisms and irregular pyramids

John built a triangular prism. Then he pushed gently on the shape, transforming two of the rectangular faces into parallelograms without right angles. "Look!" he said excitedly. "I squished the prism. Does this shape have a name? Is it still a prism?"

I had anticipated this opportunity to help students expand their understanding of prisms. We had a class discussion about the difference between a *right* prism, in which the lateral faces are perpendicular to the bases, and an *oblique* prism. Other students experimented with "squishing" other kinds of prisms by pushing them in one or more directions (see fig. 6).

The straws and connectors also allowed students to expand their understanding of pyramids. Using the paper model, students had built only regular pyramids, in which the base is a regular polygon with equal sides and equal angles and the vertex is directly over the center of the base. Using straws, students were able to build a greater variety of pyramids (see fig. 7). Students also tested the strength of pyramids and found that these shapes are stable and cannot be transformed or "squished" like the prisms.

Structured investigations and problem solving

After a period of free exploration, I posed the following problem for students to investigate in small groups: How many different rectangular prisms can you build using straws of two different lengths: 3 inches and 7 inches?

In one group, students quickly set to work building the two cubes as $3 \times 3 \times 3$ inches and $7 \times 7 \times 7$ inches. I introduced the term *similar*, meaning the same shape but a different size. Mel was working on another prism with the dimensions $7 \times 3 \times 3$ inches. She asked, "What do you mean by 'different'? Would this prism be different if I turned it around so that it is tall?" We then reviewed the term *congruent*, which students had encountered earlier in their study of plane geometry. After some thought, Sara Beth said, "Well, we could have a big square on the bottom, with the shorter edges coming up. That would be different." She set to work building the $7 \times 7 \times 3$ solution.

I encouraged students to record their solutions, using drawings, words, and numbers. Some students were very interested in drawing two-dimensional representations of the three-dimensional models and spent a lot of time practicing this skill. I listened to students with more drawing experience give tips to their classmates, such as "Draw the closest face first. Then make lines going back."

During choice time, a number of students were interested in exploring the following, more challenging problems:

- How many different rectangular prisms can you build using straws of three different lengths: 3 inches, 5 inches, and 7 inches? (There are ten possibilities.)
- How many different triangular pyramids can you build using straws of two different lengths: 5 inches and 7 inches? (Students were surprised to find nine different arrangements of straws, including both regular and irregular pyramids. This problem stimulated animated discussion about congruency and symmetry.)

Counting Faces, Edges, and Vertices: Connecting Geometry and Algebra

"How many faces does a cube have?" My students were confident that a cube has six faces. They had been using the terms *face*, *vertex*, and *edge* to describe shapes. Now I wanted them to think about how the structure of a figure relates to the number of its faces, vertices, and edges. I asked, "How many

faces does a prism have?" I was pleased when students responded, "It depends on the kind of prism."

I knew that counting faces, vertices, and edges of three-dimensional shapes would provide rich opportunities for exploring algebraic patterns. As *Principles and Standards* states, "Students should have the opportunity to apply geometric ideas and relationships to other areas of mathematics. . . . In algebra, students in grades 3–5 often work with geometric patterns to explore patterns and functions" (NCTM 2000, p. 169).

I gave the students charts to help them record their information (see **fig. 8**). As students counted and recorded numbers, many of them expressed excitement as they observed emerging patterns. With some guided questions, students were able to connect the numeric patterns with the geometric structure of prisms as follows.

Student 1. The number of vertices for a prism is even.

Teacher. Does this make sense? Why would the number of vertices of a prism be an even number?

Student 2. Well, for a triangular prism, there are three points at the bottom and three more at the top. That's double. It's an even number.

*Teacher.* Would this be true for any prism? What if the prism had five sides on the base?

Student 1. Then there would be five points on the bottom and five more on the top. It's always double.

To help students generalize the relationship between the number of sides on the base of a prism and the number of faces, vertices, and edges, I asked them to visualize a prism with ten sides on the base. I said, "Try to picture a decagonal prism in your mind, without building a model. What can you say about the number of faces, vertices, and edges?" Many students answered this question successfully, either by visualizing the prism, analyzing the numeric patterns, or both.

With a group of interested students, I introduced the idea of using the variable n to represent the number of sides on the base of a prism. Students were able to generate the functions: Faces $= n + 2$, Vertices $= n * 2$, Edges $= n * 3$.

FIGURE 6 Right and oblique prisms
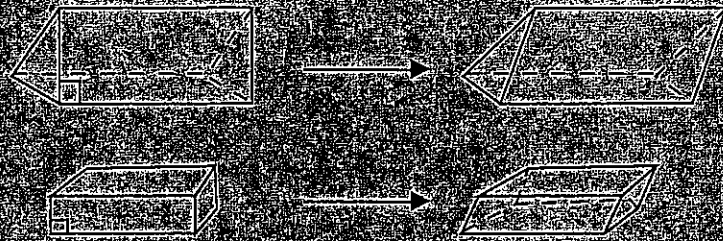
In a similar way, we counted faces, vertices, and edges of different pyramids and analyzed patterns. After their experience with prisms, many students were able to analyze and generalize patterns for pyramids independently; for example, using the variable n to represent the number of sides on the base of a pyramid: Faces $= n + 1$, Vertices $= n + 1$, Edges $= n * 2$.

Describing Prisms and Pyramids, Continued

Throughout the week, the students continued to add to their list of descriptions on the charts labeled "Properties of Prisms" and "Properties of Pyramids." A few students attempted to develop definitions by using minimal facts, a practice characteristic of the formal-deduction level of thinking. For example, Noah read from his journal: "A prism is a shape elongated with its edges forming rectangles to another shape, the exact same one as the starting shape that was elongated." In a class discussion, we considered Noah's definition. One student suggested changing the word "rectangles" to "parallelograms" so that the definition would include oblique prisms. Toward the end of the week, I shared some textbook definitions of *prism* and *pyramid* with the class. Students were particularly intrigued by the following definition: "A prism is a three-dimensional shape whose cross-sections, cut parallel to an end face, are the same shape as the end faces." Students realized that this definition also would include cylinders, so we added a statement that the end faces are polygons.

In addition to participating in whole-class and

FIGURE 7 Regular and irregular pyramids
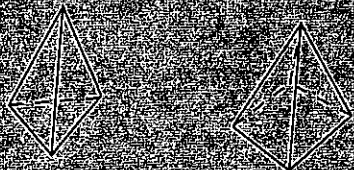
Regular pyramids

Irregular pyramids

Patterns in prisms

Name of Shape	Number of Sides for the Base	Number of Faces	Number of Vertices	Number of Edges
Triangular prism	3	(5)	(6)	(9)
Rectangular prism	4	(6)	(8)	(12)
Pentagonal prism	5	(7)	(10)	(15)
Hexagonal prism	6	(8)	(12)	(18)
n-gonal prism	n	$(n + 2)$	$(n \cdot 2)$	$(n \cdot 3)$

small-group discussions, students kept individual journals in which they recorded descriptions, drawings, and solutions to problems. During the week, I observed students moving through different levels of geometric thinking. At the beginning of the week, Claire wrote in her journal: "A prism is a shape stretched out. A pyramid is a shape with a pointy top." She was describing the holistic appearance of these shapes and thinking at the visual level. After building models with straws and participating in class discussions, Claire added to her descriptions: "A prism has parallel edges and parallel bases. A pyramid has triangles for sides that meet at the top and form a vertex." Now she was using more precise vocabulary to describe specific properties of shapes, showing descriptive-level thinking. After looking for patterns in a prism's number of faces, vertices, and edges, Claire wrote this generalization: "For a prism, the number of faces is two more than the number of sides on the base because you have a face for every side and then you add the two bases. For a pyramid, the number of faces is just one more than the number of sides on the base."

In reading the students' journal entries, I noticed how they made personal sense of ideas we had discussed in class. For example, Laura made the following journal entry after a class discussion about the distinction between a base and a lateral face: "Pyramids always have a vertex at the top, no matter how many faces. The base is always the shape that is not a triangle, unless it is a triangular pyramid, and in that case any face can be the base. Prisms always have two bases. If it is a rectangular prism, it doesn't matter which side is the base because all of the faces are rectangles."

Conclusion

"When can we build again?" By the end of the week, my students had all become successful model builders and they were eager to continue exploring with the straws and connectors. These simple materials had allowed students to construct knowledge about three-dimensional shapes in an active way. As they built, students also were engaged in talking about their models, so they learned and used new vocabulary in a meaningful context. Instead of simply memorizing definitions, students were developing precise ways to describe and classify shapes based on their own observations and experiences.

References

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, Va.: NCTM, 2000.

Van Hiele, Pierre M. "Developing Geometric Thinking through Activities That Begin with Play." *Teaching Children Mathematics* 5 (February 1999): 310–16. ▲

The Platonic Solids

Five special polyhedra are called the **Platonic solids**, in honor of Plato, who thought of these solid shapes as associated with earth, fire, water, air, and the whole universe. These are the five Platonic solids:

Tetrahedron is made of 4 equilateral triangles, with 3 triangles coming together at each corner.

Cube is made of 6 squares, with 3 squares coming together at each corner.

Octahedron is made of 8 equilateral triangles, with 4 triangles coming together at each corner.

Dodecahedron is made of 12 regular pentagons, with 3 pentagons coming together at each corner.

Icosahedron is made of 20 equilateral triangles, with 5 triangles coming together at each corner.

The Platonic solids are special because each one is made of only one kind of polygon, and the same number of polygons come together at each corner. The names of the Platonic solids make sense because *hedron* comes from the Greek for *faces*, while *tetra, octa, dodeca,* and *icosa* mean 4, 8, 12, and 20, respectively. So for example, *icosahedron* means *20 faces*, which makes sense because an icosahedron is made out of 20 triangles.

While cubes are commonly found in daily life (such as boxes and ice cubes), the other Platonic solids are not commonly seen. However, these shapes do occur in nature occasionally. For example, the mineral pyrite can form a crystal in the shape of a dodecahedron (this crystal is often called a pyritohedron).

tetrahedron

octahedron

cube

dodecahedron

icosahedron