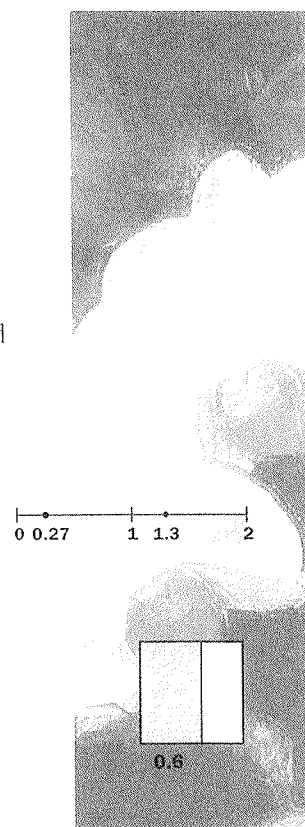


Decimals

Decimals extend whole number place value to represent quantities that are between whole numbers. Even though decimals extend whole numbers, students need to learn special conventions for reading and writing decimals. Understanding decimals includes understanding that there is always a decimal number between any two decimals.

KEY CONCEPTS

- *Decimals represent specific quantities.*
For example, 4.75 represents a unique amount. This amount is more meaningful when the unit is known, e.g., 4.75 grams, 4.75 meters, 4.75 dollars.
- *Decimals are rational numbers, the result of dividing two numbers.*
For example, 2 kg of candies shared by 8 children gives 0.25 kg per 1 child. The decimal 0.125 is the amount equal to 125 divided by 1000, or 1 divided by 8.
- *Decimals are a type of fraction, with denominators that are powers of ten.*
The denominator of a decimal is known by counting the number of places from the decimal point to the right-most number. The value of each place is one-tenth of the value of the place to its left.
- *A decimal amount can be expressed in different place value terms.*
For example, 2.4 is equivalent to "24 tenths" or "2 and four tenths" and 4.75 grams is equivalent to "475 hundredth grams" or "4 and 75 hundredth grams." Understanding such equivalencies is related to recognizing a decimal as a single quantity, rather than as a collection of variously sized parts, and is good preparation for multiplication and division by decimals.
- *Additional decimal places can be used to report more precise measurements.*
For example, a pencil might be measured as 19 cm, 18.6 cm, or even 18.58 cm. Places further to the right of the decimal point represent ever smaller sub-units.



- *Decimals extend the base ten number system.*

As with whole numbers, a separate value can be given to each digit in a decimal number. The value of an individual digit depends on its place in relation to the ones place. Count the number of places to the left of the ones place to determine which power of ten to multiply the digit by. Conversely, count the number of places to the right of the ones place to decide which power of ten to divide the digit by. Two places to the left of the ones place is the hundreds place. Two places to the right of the ones place is the hundredths place. The value of the entire number is the sum of the values of the individual digits.

CONNECTIONS

An understanding of decimal place value develops insight into the properties of numbers, specifically the density of numbers. The fact that a number can always be found that is between two given numbers intrigues elementary students and is preparation for higher math.

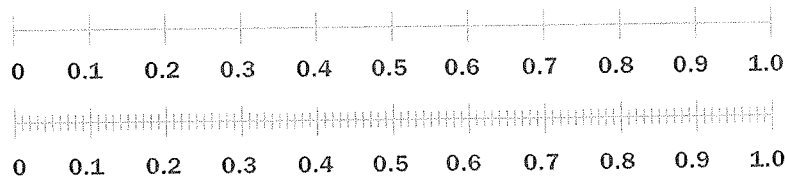
Students should be encouraged to draw on the meaning of operations with whole numbers to make sense of the operations with decimals. A weak understanding of decimal numbers will interfere with students' ability to make these connections.

TEACHING TIPS

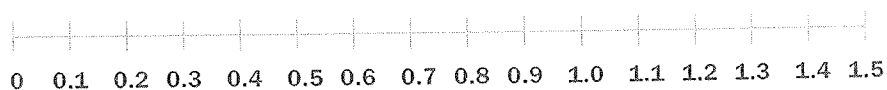
Developing Students' Understanding of Decimals

- *Emphasize that a decimal is a specific amount, with a location on a number line.*

Have students locate decimals on various number lines, using increasingly precise scales depending on the number of decimal places. Experiences with metric measurement also reinforce decimals as quantities that can represent measurements.



Have students estimate the location of numbers on a number line, including decimals such as 0.3, 0.32, 0.95, or 1.3.

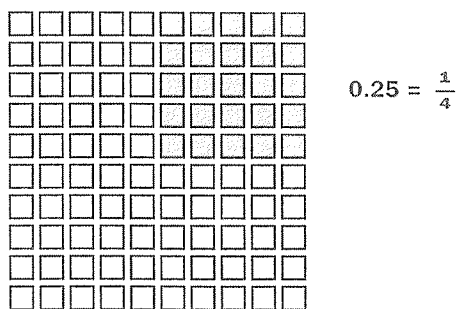


- *Discuss decimals as a fractional amount with a denominator that is a power of 10.*

Draw connections between a decimal and an equivalent fraction, for example, 2.75 and $2\frac{75}{100}$ or $2\frac{3}{4}$. This will reinforce the concept of the denominator of a decimal and link decimal understanding to prior understanding of fractions.

- *Establish and learn equivalencies to well known benchmark fractions.*

For example, $0.5 = \frac{1}{2}$; $0.25 = \frac{1}{4}$; 0.3 is near $\frac{1}{3}$.



- *Show decimals as a part of a whole as one model.*

Models involving shading 100-grids or using base ten blocks help develop the connection between decimal places and powers of 10, and also fractions. Overuse of these models may under-emphasize the fact that a decimal is a single quantity because of two-part counting of the total and of the shaded part. When using models, point out that the choice of what to designate as 1 whole unit must be established, and the value of a tenth, etc., is relative to this referent unit. When using metric measure, the whole is the unit being used. On a number line, the whole or unit is the distance from zero to one.

- *Define "decimal point."*

A decimal point signals that whole numbers are to the left, decimal parts to the right. Students need to know that "4" and "4." are equal amounts. "Lining up the decimal point to add or subtract" keeps the whole numbers together and corresponding decimal places together.

- *Give experiences making comparisons with decimals.*

Have students order several decimals with different denominators (tenths, hundredths, thousandths). This can be done by comparing places, by using or visualizing a number line or, if necessary, by changing all of the decimals to the same number of right-hand places. For example, a student who is developing a strong concept of decimals should be able to order 3.8, 3.085, 3.08, and 3.28 without difficulty. Other students may "trade" to have each quantity given in thousandth units: 3.800, 3.085, 3.080, 3.280. Such trading can be explained through the use of number lines with increasingly precise scales and also with models or shading of 100-grids.

ASSESSMENT ITEMS

Item	Percent correct	
	Grade 7	Grade 11
1. What number is GREATEST? (0.36, 0.058, 0.375, or 0.4*)	47	77
2. Which number is between .03 and .04? (correct response from 4 choices: .035)	35	73
3. Write as a fraction: .037 (correct response from 4 choices: $\frac{37}{1000}$)	48	58

(1986 NAEP, Grades 7, 11)

4. Which of the following is closest to 15 seconds?

Response choices	Percent choosing the response	
	Grade 4	Grade 8
14.1 seconds	16	2
14.7 seconds	2	<1
14.9 seconds*	63	92
15.2 seconds	18	5

(1992 NAEP, Grades 4, 8)

LEARNING PITFALLS

misreading decimals - Students sometimes misapply the methods of reading whole-number place value names to reading decimals. For example, they may think that because hundreds are the third whole-number place, hundredths should also be the third place in decimals. The first place is problematic because there are no "oneths."

treating decimals like whole numbers - Students may ignore any fractional reference to a whole and treat a decimal like a whole number. For example, when shading on a 100-grid a student may simply count 35 squares to show 0.35, or when locating 0.3 on a number line a student may simply count off 3 marks from zero, unless also faced with more challenging questions, such as "next shade 0.351." The resulting lack of a decimal concept is likely to cause later calculation difficulties.

treating decimals as collections of digits - Students who have focused on place value names, and who have prior experience with base ten blocks as representations of whole numbers, may use the blocks to count out a decimal amount as a collection of units, without making any reference to the decimal number as a specific quantity that represents a part of a whole. This prevents them from making sense of operations and decimals used in real-world situations.

thinking that decimals are negative numbers - Sometimes students think that anything less than one is negative. For example, they think of 0.3 as less than zero on a number line. The decimal point may be confused with zero on the number line.

Reading, Writing, and Discussing Decimals

- *Use various models to represent decimals and make the connection to the written decimal form.*

Models can help develop understanding and lead to correct use of the "line up the decimal points to add or subtract" rule, and correct ordering of decimals.

- *Emphasize the denominator of a decimal amount when naming decimals.*

Having students read 4.08 sec as "4 and 8 hundredths seconds" as opposed to "4 point zero 8," emphasizes the value of the non-whole portion. Thus, 4.08 sec is more likely to be recognized as faster, or less, than 4.1 sec.

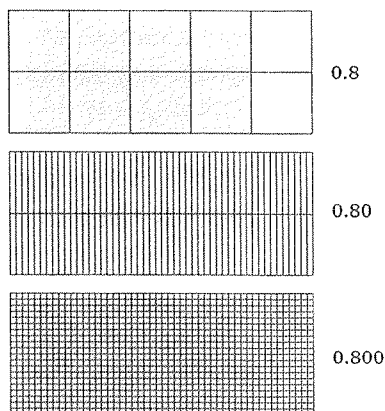
- *Show the denominator first when writing a decimal heard orally.*

Students can write "0. _ _ _" in preparation for writing "five thousandths." Because decimal denominators are not seen but are inferred from the number of places, this method draws attention to the denominator.

- *Discuss the similarities and differences between whole numbers and decimals.*

Whole numbers have a base ten place value system and are read in a manner that states each place's value. For example, 234 would be read, "Two hundred thirty (three tens) four (the ones is not stated)," giving a verbal reference to each place's value. Decimal fractions also have places which are based on tens, but a single denominator is given for the entire non-whole portion and the denominator used varies according to the number of decimal places. For example, 0.308 is "three hundred eight *thousandths*." In English, this value is never read by individual places; for example, 0.308 would *not* be read as "three tenths and eight thousandths."

- *Look at examples of equal amounts where the denominator varies according to the number of decimal places.*



Discuss how to read a set of decimals such as 0.8, 0.80, 0.800, 0.8000, and relate each amount to both a number line model and a shaded model.

LEARNING PITFALLS

ignoring decimal values - Students who read decimals as "point 8," etc., often make errors in ordering or comparing decimals. They fall back on whole-number rules and conclude that "point 8" is less than "point 25."

misapplying fractional understandings to decimals - Students who do not understand the implied denominator in the decimal system may use a system that they do know, fractions, to write decimals. For example, it is not uncommon for a student to write 5.100 when asked to write "five hundredths" as a decimal.

confusing fraction and decimal notation - Some students might write $\frac{3}{5}$ as 3.5.

Developing Computational Facility

- *Teach computation procedures and concepts after establishing fundamental concepts of decimals.*

Students will need to spend several weeks to develop proficiency with the basics of decimals, before developing facility with computation. Often, 5- or 10-minute daily activities involving comparing, ordering, or representing specific decimal quantities in a variety of ways is most effective.

- *Teach operations with decimals through examples.*

Students should continue to connect the use of algorithms with actual situations and with the meaning of the operation. For example, "4 children each carry 3.2 kg..." (multiply) "a rectangle measures 4.0 m by 3.2 m," (multiply) "they mixed 3.5 liters with 4.8 liters." (add)

- *Emphasize the logic behind the "line up the decimal points to add or subtract" rule.*

Once students understand the value of individual places, it should be understood that tenths must be added to tenths, and hundredths with hundredths.

- *Teach estimation, before algorithms, so that computation problems can be checked for reasonable answers.*

Have students estimate the results of multiplication and division problems by rounding to the nearest whole or half whenever possible. For example, 3.24×0.7 can become "3 x 0.5, or about 1.5, but a bit more because both numbers were rounded down." Rounded numbers may allow the student to think of such a problem in terms of the meaning of multiplication; for example, "about 3 times as much as 0.5 pounds," or "about 3 packages, each weighing close to 0.5 pounds."

- *Teach students to estimate the results of addition and subtraction problems.*

It is often sufficient to simply "shorten" each number to the same place when estimating. For example, $4.065 + 0.533 + 1.2$ can be thought of as

$$4.0 + 0.5 + 1.2 \approx 5.7.$$

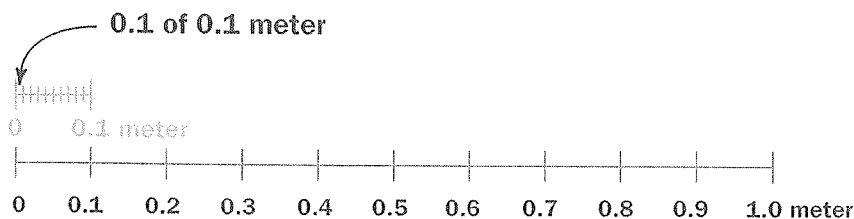
Estimation leads not only to more calculation success, but also to a stronger grasp of the relative value of different places.

- *Limit the use of money as an example.*

Money may be helpful in initial decimal division lessons. But most students think of dimes and pennies as separate whole number amounts, and so a decimal money amount is not viewed as a single decimal quantity. Also, money cannot usually be continued to be subdivided into smaller sub-units past a hundredth dollar.

- *Show decimal multiplication in terms of finding a part of a part of a unit.*

Start by examining 0.1×0.1 meter as "one tenth of one tenth meter." This part of 0.1 meter is 0.01 meter, "one hundredth of a meter." Once students know that $0.1 \times 0.1 = 0.01$, problems such as "What is 1 tenth of 4 tenths?" can be interpreted as being "four hundredths, or 0.04." Similarly, 0.2×0.9 is eighteen hundredths, or 0.18.



- *In multiplying decimals, develop the rationale for counting the number of decimal places.*

One way to explain this algorithm is to convert a decimal multiplication problem to a fraction problem. For example, 0.4×0.23 converts to $\frac{4}{10} \times \frac{23}{100}$. Multiplying the numerators is the same as multiplying the numbers in the decimal problem without considering the decimals ($4 \times 23 = 92$). Multiplying the denominators gives thousandths. Counting the number of decimal places also gives thousandths because there are three digits behind decimals in the two factors (0.4 and 0.23). Both methods lead to the product 0.092.

- *Develop procedures for division based on understanding of equivalent problems.*

When there is a decimal in the divisor, develop procedures for converting the problem to an equivalent problem without a decimal in the divisor. For example, when faced with the problem $0.03 \overline{)0.12}$ or $0.12 \div 0.03$, it is important to recognize that $0.12 \div 0.03 = 1.2 \div 0.3 = 12 \div 3$, and also to recognize that the problem asks "How many three hundredths are in twelve hundredths?"

Similarly, students can change a problem such as $0.3 \overline{)1.26}$ into a more recognizable form by multiplying both the divisor and dividend by 10, giving $3 \overline{)12.6}$. This method, sometimes called "clearing the divisor," is based on the fact that in a division problem, multiplying the divisor and the dividend by the same factor does not alter the quotient. Once there is a whole number divisor, the decimal in the quotient can be placed by aligning it with the decimal in the dividend.

See the chapter on Division for more decimal division tips.

ASSESSMENT ITEMS

Item	Percent correct	
	Grade 7	Grade 11
1. $6.002 + .02 + 100.4$	59	83
2. $4.3 - .53$	43	65
3. $\begin{array}{r} 7.2 \\ \times 2.5 \\ \hline \end{array}$	62	76
4. $.2 \times .4$	58	78
5. $.3 \overline{)9.06}$	52	67

(1986 NAEP Grades 7, 11)

LEARNING PITFALL

not recognizing whole numbers as decimals - Some students do not know where to put the decimal in a whole number. For example, they are confused when asked to add $4 + 0.8$ because there is no decimal point to line up, or they add $2.5 + 0.4$, giving an answer of 2.9.

Algebra & Functions Foundations

Three concepts form a solid foundation for formal study of algebra and functions. Experiences in the elementary grades with variables, functions, and equality form a basis for later concepts and help avoid misconceptions and pitfalls that hamper success.

KEY CONCEPTS

- *Variables, represented by letters, are used to represent quantities that change.*

An important use of variables is to state the relationship between 2 (or more) variables. As one variable takes various values, a related variable also changes value. Variables make it possible to express general rules of numerical relationships. For example, the total (t), including tax, for an item with a price (p) in a county with 6% sales tax can be expressed by $t = 1.06p$. This equation helps describe how the total will change as the price changes dollar by dollar and will work for any price.

Single variables can also represent *unknowns* in equations, rather than quantities that change. For example, $2(10 - x) = 14$ is an equation that can be found to be true when $x = 3$.

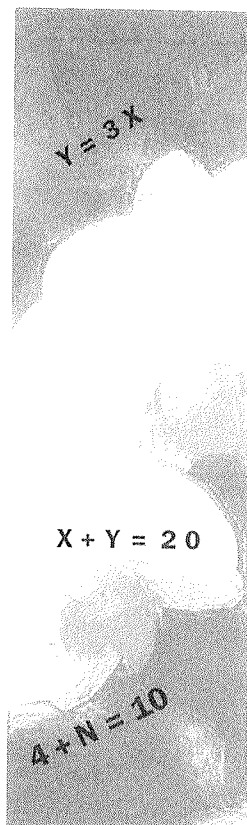
- *A function is a relationship between two variables that vary.*

One variable is the independent variable, the other the dependent variable. In the example above, the price (p) is the independent variable. Once p is selected, the value of t *depends* on the value of p in such a way that the value of t is completely determined by the value of p . Similarly, the area of a circle is determined by the chosen radius. The equation $A = \pi r^2$ shows that the area of a circle is a function of the radius. The area is always equal to the radius squared times π . As the radius varies, so does the area. An example of a function for younger children is a pricing situation involving the relationship between a price rate such as \$.90 per pound, the number of pounds of apples, and the cost of a bag of apples.

Mathematicians define a function as a relation between two sets where each value of the first set corresponds to one and only one value in the second set. In the apples example, this means that each weight of apples is paired with only one possible price.

- *Equations are statements of equality, or equal value, between two expressions.*

An equation may be an identity, as in $N + 0 = N$, or $a(b + c) = ab + ac$, which is true for all values of the variables. An equation may have a specific solution set for which it is true, as in $2x = 11$. Equations may show two different ways of representing the same quantity, as in $2(x + x + 4) = 4x + 8$. Equations may show general relationships as in $t = 1.06p$.



CONNECTIONS

Before working with formal algebra, students should have experience with variables, with functions and their representations, and with the meaning of "equal." In formal algebra students will express general numerical relationships and mathematical structures symbolically and work with the relationships in more complex ways than arithmetic methods. A mathematical structure written with symbols can be operated on. Students will learn to combine like terms, factor, and subtract the same term from both sides of an equation.

While many people do not use formal algebra or functions in their everyday work lives, the concepts embedded in algebra give people tools to consider varying quantities in various situations and to think abstractly.

TEACHING TIPS

Understanding the Foundations of Algebra and Functions

- *Give students early experiences with variables used in general relationships so that variables are not seen as only representing "unknown" or "missing" quantities.*

For example, students can list all of the possible whole-number (x,y) pairs of values when $x + y = 10$. Also, they can discuss possible values (including non-whole number values) for x and y in $5x = y$, or in $xy = 24$.

$5x = y$	
x	y
1	5
2	10
0.5	2.5
10	50

- *Provide experiences with data from a relationship that is expressed verbally, in a table, on a graph, and as a symbolic rule (equation).*

Include examination of data from science class, such as plant growth vs. time data, noting relationships such as, "It grew about $1/4$ inch per day from Day 24 to Day 35." Show the data in a table and on a coordinate graph.

Study the perimeter of squares as the length of the sides change. Note that the total perimeter depends on the length of a side and is related to 4 times the length of a side. Look at the data in a table and on a coordinate graph, and write a simple rule, "perimeter equals 4 times the side length."

- *Develop the concept of a variable by first looking at a variety of numerical replacements, and gradually discussing variables as representing a range of values.*

For example, $C = 0.25N$ can represent the cost of various numbers of 25 cent candies and can be tested for a variety of whole-number values of N . Students can discuss what range of values of N are possible for this situation.

- Give students in-depth experiences with a real-life function relationship between two quantities, and then express the relationship using variables.

For example, students can simulate having a candy store where people choose and weigh bags of candy in kilograms and multiply the weight by the price per kilogram to find out the price. A chart of weights and corresponding prices can be made, variables chosen, and the relationship represented to show that the two variables are related by multiplying by the constant cost per kilogram.

$w \cdot \$3.00 = p$	
weight (w)	price (p)
0.5 kg	\$1.50
1	3.00
4	12.00
10	30.00

- Look for a constant rate of change in the dependent variable when examining data in a table.

Notice situations where an increase of 1 for the independent variable is accompanied by a constant increase in the data for the dependent variable.

?	
x	y
0	3
1	7
2	11
3	15

This situation is important because it occurs in linear functions that have straight-line graphs. For example, this chart has an increase of 4 for the second variable for every increase of 1 for the first variable. Extend the table to predict other values. Challenge students to orally, and later symbolically, describe how to obtain the second value (y) when the first value (x) is known. This is sometimes called "What's My Rule?" and involves actually finding the function rule for a set of data.

- Emphasize that an equal sign expresses a symmetric relationship of equality between two sides of an equation.

Elementary mathematics experience may have led students to think that an equal sign signals that an "answer" is coming, or that it says "gives" or "makes." Explore number sentences such as $3 + 1 + 1 = 6 - 1$ to emphasize "is the same as." Look at equal fractions such as $\frac{1}{4} = \frac{16}{64}$ to reinforce the idea that equal quantities can be represented by numbers (or expressions) that do not look the same. Some students will also be ready to recognize that an equation remains a statement of equality when the same amount is added, or subtracted, from each side. For example, $3 + 1 + 1 + 20 = 6 - 1 + 20$ is a true equation, based on the equation above and adding 20 to both sides.

- Encourage students to write several possible equations for a situation.

For example, the cost of buying sets of party favors for n guests (based on prices in the table) might be re-presented by $15n + 15n + 30n + 40n$, or by $2(15n) + 30n + 40n$, or by $n(15 + 15 + 30 + 40)$. Notice that the first and third expressions together are an example of the distributive property.

All Party Favors On Sale Today!	
Balloons.....	15¢ each
Party Hats.....	15¢ each
Bubble Mix.....	30¢ each
Mini Puzzles	40¢ each

- *Look at specific differences between arithmetic and algebraic conventions.*

In arithmetic, when given a symbolic expression such as $4 + 8$, procedures can be followed and an answer can always be found. In algebra, abstract symbolic expressions have meaning as mathematical entities and may not be tied to any concrete problem or specific numeric answer. For example, the expression $x + 4$ names the operation of adding 4 to x , and also describes the quantity "four more than x ," for any value of x .

- *Give informal exposure to the properties of algebra while working on the concept of equality.*

Students should not only be told that $a + b = b + a$ works for all numbers a and b . They should also be asked "Do you think this is always true? Can you think of a counter-example when it would not be true?" The following properties (without their formal names at first) should be explored in such an inquiry-based manner during the elementary years:

Commutative properties of addition and multiplication:

$$a + b = b + a \text{ and } a \times b = b \times a$$

Associative properties of addition and multiplication:

$$a + (b + c) = (a + b) + c \text{ and } a \times (b \times c) = (a \times b) \times c$$

Distributive property:

$$a \times (b + c) = a \times b + a \times c$$

ASSESSMENT ITEM

1. Marlene made 6 batches of muffins. There were 24 muffins in each batch. Which of the following number sentences could be used to find the number of muffins she made?

Response choices	Percent choosing the response
	Grade 4
$6 \times \square = 24$	27
$6 + 24 = \square$	9
$6 + \square = 24$	3
$6 \times 24 = \square^*$	37

(1992 NAEP, Grade 4)

LEARNING PITFALLS

applying place value understanding to algebra expressions such as $4x$ - Students sometimes evaluate $4x$ as 42 when given that $x = 2$.

adding rather than multiplying for expressions such as xy - Students may add the values of the two variables rather than multiplying.

applying mental math problem solving strategies when algebraically representing a situation -

For example, "Carol has some cards, but Joe has 4 more than Carol. If Joe has 30, how many does Carol have?" In arithmetic, students learn to subtract in this "4 more" situation. But the structure and word order of the situation should be linked to $c + 4 = 30$, rather than to $c = 30 - 4$.

adding coefficients and numbers - For example, "If $x = 10$ crayons and the number of crayons $= 4 + 3x$, how many crayons are there?" may mistakenly be solved as $7x = 70$ crayons.

Reading, Writing, and Discussing Algebra

- *Distinguish between the use of letters as abbreviations for units of measurement and their use as variables.*

For example, $5m$ means 5 meters in arithmetic. In algebra, if m stands for meters, it would mean 5 times the *number* of meters.

- *Explain that any letter may be chosen for a variable, and that choices are made so that the variable is easy to recognize, but any letter may be used.*

For example, d is often used to represent a distance, but x could be used. Encourage students to write down a "key" such as " $d =$ distance in km" or " $x =$ the number of pounds."

- *Define variables with specific phrases, such as "Let t equal the number of teachers."*

Phrases such as "let b equal boys" mask the numeric meaning of a variable. The variable b might represent the weight of the boys in pounds, or the money earned by the boys, or simply the number of boys.

- *Reinforce the meaning of the equal sign.*

It can be read as "equals" or "is the same as." Include experiences with equations with the variable isolated on either the left or the right, such as $x = 25(100 - 1)$ and also equations such as $4^2 = x + 2$, and $x = x$. Let students invent "fancy" ways of expressing a quantity. For example, after starting with $16 = 16$ a student might write $16 = 4 \cdot 4 = (10 - 6)2^2$.