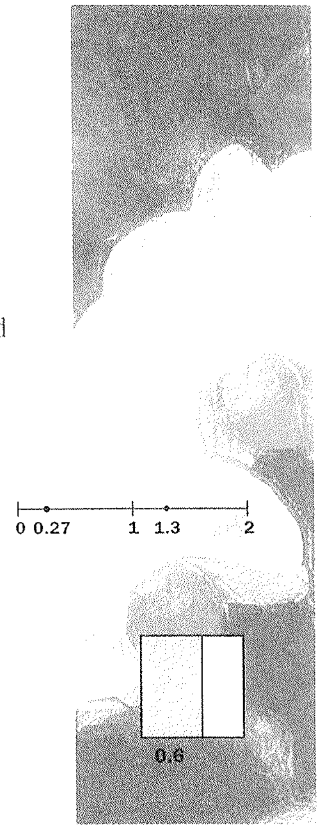
Decimals

Decimals extend whole number place value to represent quantities that are between whole numbers. Even though decimals extend whole numbers, students need to learn special conventions for reading and writing decimals. Understanding decimals includes understanding that there is always a decimal number between any two decimals.



KEY CONCEPTS

- *Decimals represent specific quantities.*

  For example, 4.75 represents a unique amount. This amount is more meaningful when the unit is known, e.g., 4.75 grams, 4.75 meters, 4.75 dollars.

- *Decimals are rational numbers, the result of dividing two numbers.*

  For example, 2 kg of candies shared by 8 children gives 0.25 kg per 1 child. The decimal 0.125 is the amount equal to 125 divided by 1000, or 1 divided by 8.

- *Decimals are a type of fraction, with denominators that are powers of ten.*

  The denominator of a decimal is known by counting the number of places from the decimal point to the right-most number. The value of each place is one-tenth of the value of the place to its left.

- *A decimal amount can be expressed in different place value terms.*

  For example, 2.4 is equivalent to "24 tenths" or "2 and four tenths" and 4.75 grams is equivalent to "475 hundredth grams" or "4 and 75 hundredth grams." Understanding such equivalencies is related to recognizing a decimal as a single quantity, rather than as a collection of variously sized parts, and is good preparation for multiplication and division by decimals.

- *Additional decimal places can be used to report more precise measurements.*

  For example, a pencil might be measured as 19 cm, 18.6 cm, or even 18.58 cm. Places further to the right of the decimal point represent ever smaller sub-units.

- ***Decimals extend the base ten number system.***

  As with whole numbers, a separate value can be given to each digit in a decimal number. The value of an individual digit depends on its place in relation to the ones place. Count the number of places to the left of the ones place to determine which power of ten to multiply the digit by. Conversely, count the number of places to the right of the ones place to decide which power of ten to divide the digit by. Two places to the left of the ones place is the hundreds place. Two places to the right of the ones place is the hundredths place. The value of the entire number is the sum of the values of the individual digits.

CONNECTIONS

An understanding of decimal place value develops insight into the properties of numbers, specifically the density of numbers. The fact that a number can always be found that is between two given numbers intrigues elementary students and is preparation for higher math.

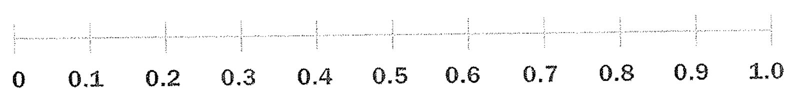
Students should be encouraged to draw on the meaning of operations with whole numbers to make sense of the operations with decimals. A weak understanding of decimal numbers will interfere with students' ability to make these connections.

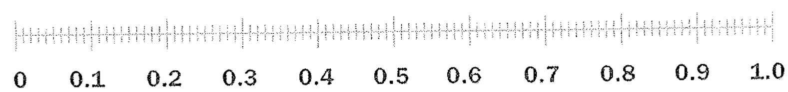
TEACHING TIPS

Developing Students' Understanding of Decimals

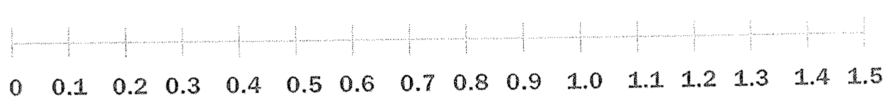
- ***Emphasize that a decimal is a specific amount, with a location on a number line.***

  Have students locate decimals on various number lines, using increasingly precise scales depending on the number of decimal places. Experiences with metric measurement also reinforce decimals as quantities that can represent measurements.





Have students estimate the location of numbers on a number line, including decimals such as 0.3, 0.32, 0.95, or 1.3.

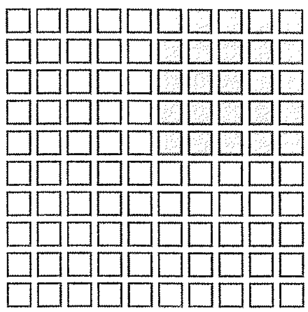


- *Discuss decimals as a fractional amount with a denominator that is a power of 10.*

  Draw connections between a decimal and an equivalent fraction, for example, 2.75 and $2\frac{75}{100}$ or $2\frac{3}{4}$. This will reinforce the concept of the denominator of a decimal and link decimal understanding to prior understanding of fractions.

- *Establish and learn equivalencies to well known benchmark fractions.*

  For example, $0.5 = \frac{1}{2}$; $0.25 = \frac{1}{4}$; 0.3 is near $\frac{1}{3}$.



$0.25 = \frac{1}{4}$

- *Show decimals as a part of a whole as one model.*

  Models involving shading 100-grids or using base ten blocks help develop the connection between decimal places and powers of 10, and also fractions. Overuse of these models may under-emphasize the fact that a decimal is a single quantity because of two-part counting of the total and of the shaded part. When using models, point out that the choice of what to designate as 1 whole unit must be established, and the value of a tenth, etc., is relative to this referent unit. When using metric measure, the whole is the unit being used. On a number line, the whole or unit is the distance from zero to one.

- *Define "decimal point."*

  A decimal point signals that whole numbers are to the left, decimal parts to the right. Students need to know that "4" and "4." are equal amounts. "Lining up the decimal point to add or subtract" keeps the whole numbers together and corresponding decimal places together.

- *Give experiences making comparisons with decimals.*

  Have students order several decimals with different denominators (tenths, hundredths, thousandths). This can be done by comparing places, by using or visualizing a number line or, if necessary, by changing all of the decimals to the same number of right-hand places. For example, a student who is developing a strong concept of decimals should be able to order 3.8, 3.085, 3.08, and 3.28 without difficulty. Other students may "trade" to have each quantity given in thousandth units: 3.800, 3.085, 3.080, 3.280. Such trading can be explained through the use of number lines with increasingly precise scales and also with models or shading of 100-grids.

ASSESSMENT ITEMS

Item	Percent correct	
	Grade 7	Grade 11
1. What number is GREATEST? (0.36, 0.058, 0.375, or 0.4*)	47	77
2. Which number is between .03 and .04? (correct response from 4 choices: .035)	35	73
3. Write as a fraction: .037 (correct response from 4 choices: $\frac{37}{1000}$)	48	58

(1986 NAEP, Grades 7, 11)

4. Which of the following is closest to 15 seconds?

Response choices	Percent choosing the response	
	Grade 4	Grade 8
14.1 seconds	16	2
14.7 seconds	2	<1
14.9 seconds*	63	92
15.2 seconds	18	5

(1992 NAEP, Grades 4, 8)

LEARNING PITFALLS

misreading decimals - Students sometimes misapply the methods of reading whole-number place value names to reading decimals. For example, they may think that because hundreds are the third whole-number place, hundredths should also be the third place in decimals. The first place is problematic because there are no "oneths."

treating decimals like whole numbers - Students may ignore any fractional reference to a whole and treat a decimal like a whole number. For example, when shading on a 100-grid a student may simply count 35 squares to show 0.35, or when locating 0.3 on a number line a student may simply count off 3 marks from zero, unless also faced with more challenging questions, such as "next shade 0.351." The resulting lack of a decimal concept is likely to cause later calculation difficulties.

treating decimals as collections of digits - Students who have focused on place value names, and who have prior experience with base ten blocks as representations of whole numbers, may use the blocks to count out a decimal amount as a collection of units, without making any reference to the decimal number as a specific quantity that represents a part of a whole. This prevents them from making sense of operations and decimals used in real-world situations.

thinking that decimals are negative numbers - Sometimes students think that anything less than one is negative. For example, they think of 0.3 as less than zero on a number line. The decimal point may be confused with zero on the number line.

Reading, Writing, and Discussing Decimals

- *Use various models to represent decimals and make the connection to the written decimal form.*

  Models can help develop understanding and lead to correct use of the "line up the decimal points to add or subtract" rule, and correct ordering of decimals.

- *Emphasize the denominator of a decimal amount when naming decimals.*

  Having students read 4.08 sec as "4 and 8 hundredths seconds" as opposed to "4 point zero 8," emphasizes the value of the non-whole portion. Thus, 4.08 sec is more likely to be recognized as faster, or less, than 4.1 sec.

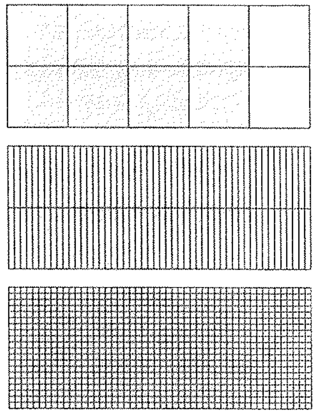
- *Show the denominator first when writing a decimal heard orally.*

  Students can write "0._ _ _ " in preparation for writing "five thousandths." Because decimal denominators are not seen but are inferred from the number of places, this method draws attention to the denominator.

- *Discuss the similarities and differences between whole numbers and decimals.*

  Whole numbers have a base ten place value system and are read in a manner that states each place's value. For example, 234 would be read, "Two hundred thirty (three tens) four (the ones is not stated)," giving a verbal reference to each place's value. Decimal fractions also have places which are based on tens, but a single denominator is given for the entire non-whole portion and the denominator used varies according to the number of decimal places. For example, 0.308 is "three hundred eight *thousandths.*" In English, this value is never read by individual places; for example, 0.308 would *not* be read as "three tenths and eight thousandths."

- *Look at examples of equal amounts where the denominator varies according to the number of decimal places.*



0.8

0.80

0.800

Discuss how to read a set of decimals such as 0.8, 0.80, 0.800, 0.8000, and relate each amount to both a number line model and a shaded model.

LEARNING PITFALLS

ignoring decimal values - Students who read decimals as "point 8," etc., often make errors in ordering or comparing decimals. They fall back on whole-number rules and conclude that "point 8" is less than "point 25."

misapplying fractional understandings to decimals - Students who do not understand the implied denominator in the decimal system may use a system that they do know, fractions, to write decimals. For example, it is not uncommon for a student to write 5.100 when asked to write "five hundredths" as a decimal.

confusing fraction and decimal notation - Some students might write $\frac{3}{5}$ as 3.5.

Developing Computational Facility

- *Teach computation procedures and concepts after establishing fundamental concepts of decimals.*

  Students will need to spend several weeks to develop proficiency with the basics of decimals, before developing facility with computation. Often, 5- or 10-minute daily activities involving comparing, ordering, or representing specific decimal quantities in a variety of ways is most effective.

- *Teach operations with decimals through examples.*

  Students should continue to connect the use of algorithms with actual situations and with the meaning of the operation. For example, "4 children each carry 3.2 kg...," (multiply) "a rectangle measures 4.0 m by 3.2 m," (multiply) "they mixed 3.5 liters with 4.8 liters." (add)

- ***Emphasize the logic behind the "line up the decimal points to add or subtract" rule.***

  Once students understand the value of individual places, it should be understood that tenths must be added to tenths, and hundredths with hundredths.

- ***Teach estimation, before algorithms, so that computation problems can be checked for reasonable answers.***

  Have students estimate the results of multiplication and division problems by rounding to the nearest whole or half whenever possible. For example, 3.24 x 0.7 can become "3 x 0.5, or about 1.5, but a bit more because both numbers were rounded down." Rounded numbers may allow the student to think of such a problem in terms of the meaning of multiplication; for example, "about 3 times as much as 0.5 pounds," or "about 3 packages, each weighing close to 0.5 pounds."

- ***Teach students to estimate the results of addition and subtraction problems.***

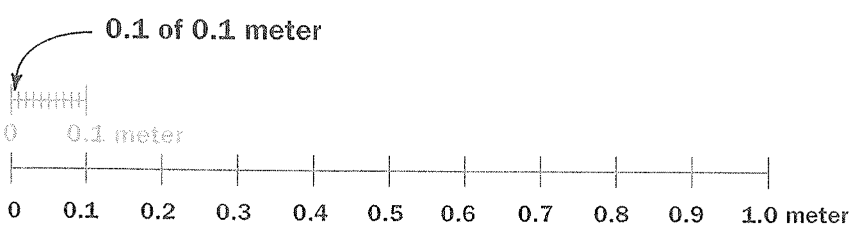
  It is often sufficient to simply "shorten" each number to the same place when estimating. For example, $4.065 + 0.533 + 1.2$ can be thought of as $4.0 + 0.5 + 1.2 \approx 5.7$.
  Estimation leads not only to more calculation success, but also to a stronger grasp of the relative value of different places.

- ***Limit the use of money as an example.***

  Money may be helpful in initial decimal division lessons. But most students think of dimes and pennies as separate whole number amounts, and so a decimal money amount is not viewed as a single decimal quantity. Also, money cannot usually be continued to be subdivided into smaller sub-units past a hundredth dollar.

- ***Show decimal multiplication in terms of finding a part of a part of a unit.***

  Start by examining 0.1 x 0.1 meter as "one tenth of one tenth meter." This part of 0.1 meter is 0.01 meter, "one hundredth of a meter." Once students know that 0.1 x 0.1 = 0.01, problems such as "What is 1 tenth of 4 tenths?" can be interpreted as being "four hundredths, or 0.04. Similarly, 0.2 x 0.9" is eighteen hundredths, or 0.18.



- ***In multiplying decimals, develop the rationale for counting the number of decimal places.***

  One way to explain this algorithm is to convert a decimal multiplication problem to a fraction problem. For example, 0.4 x 0.23 converts to $\frac{4}{10} \times \frac{23}{100}$. Multiplying the numerators is the same as multiplying the numbers in the decimal problem without considering the decimals (4 x 23 = 92). Multiplying the denominators gives thousandths. Counting the number of decimal places also gives thousandths because there are three digits behind decimals in the two factors (0.4 and 0.23). Both methods lead to the product 0.092.

- ***Develop procedures for division based on understanding of equivalent problems.***

  When there is a decimal in the divisor, develop procedures for converting the problem to an equivalent problem without a decimal in the divisor. For example, when faced with the problem. $0.03\overline{)0.12}$ or 0.12 ÷ 0.03, it is important to recognize that 0.12 ÷ 0.03 = 1.2 ÷ 0.3 = 12 ÷ 3, and also to recognize that the problem asks "How many three hundredths are in twelve hundredths?"

  Similarly, students can change a problem such as $0.3\overline{)1.26}$ into a more recognizable form by multiplying both the divisor and dividend by 10, giving $3\overline{)12.6}$. This method, sometimes called "clearing the divisor," is based on the fact that in a division problem, multiplying the divisor and the dividend by the same factor does not alter the quotient. Once there is a whole number divisor, the decimal in the quotient can be placed by aligning it with the decimal in the dividend.

  See the chapter on Division for more decimal division tips.

ASSESSMENT ITEMS

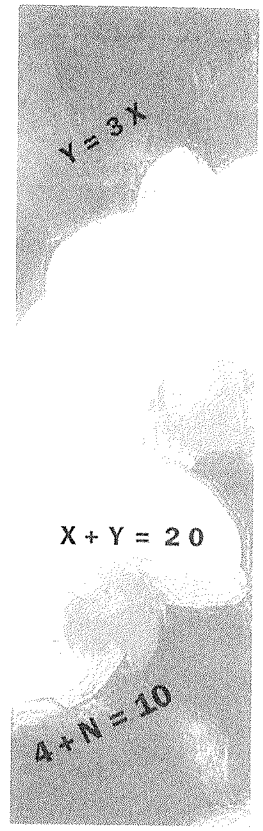
Item	Percent correct	
	Grade 7	Grade 11
1. 6.002 + .02 + 100.4	59	83
2. 4.3 – .53	43	65
3. $\begin{array}{r} 7.2 \\ \times\ 2.5 \\ \hline \end{array}$	62	76
4. .2 x .4	58	78
5. $.3\overline{)9.06}$	52	67

(1986 NAEP Grades 7, 11)

LEARNING PITFALL

not recognizing whole numbers as decimals - Some students do not know where to put the decimal in a whole number. For example, they are confused when asked to add 4 + 0.8 because there is no decimal point to line up, or they add 2.5 + 0.4, giving an answer of 2.9.

Algebra & Functions Foundations



Three concepts form a solid foundation for formal study of algebra and functions. Experiences in the elementary grades with variables, functions, and equality form a basis for later concepts and help avoid misconceptions and pitfalls that hamper success.

KEY CONCEPTS

- ***Variables, represented by letters, are used to represent quantities that change.***

  An important use of variables is to state the relationship between 2 (or more) variables. As one variable takes various values, a related variable also changes value. Variables make it possible to express general rules of numerical relationships. For example, the total (t), including tax, for an item with a price (p) in a county with 6% sales tax can be expressed by $t = 1.06p$. This equation helps describe how the total will change as the price changes dollar by dollar and will work for any price.

  Single variables can also represent *unknowns* in equations, rather than quantities that change. For example, $2(10 - x) = 14$ is an equation that can be found to be true when $x = 3$.

- ***A function is a relationship between two variables that vary.***

  One variable is the independent variable, the other the dependent variable. In the example above, the price (p) is the independent variable. Once p is selected, the value of t *depends* on the value of p in such a way that the value of t is completely determined by the value of p. Similarly, the area of a circle is determined by the chosen radius. The equation $A = \pi r^2$ shows that the area of a circle is a function of the radius. The area is always equal to the radius squared times π. As the radius varies, so does the area. An example of a function for younger children is a pricing situation involving the relationship between a price rate such as $.90 per pound, the number of pounds of apples, and the cost of a bag of apples.

  Mathematicians define a function as a relation between two sets where each value of the first set corresponds to one and only one value in the second set. In the apples example, this means that each weight of apples is paired with only one possible price.

- ***Equations are statements of equality, or equal value, between two expressions.***

  An equation may be an identity, as in $N + 0 = N$, or $a(b + c) = ab + ac$, which is true for all values of the variables. An equation may have a specific solution set for which it is true, as in $2x = 11$. Equations may show two different ways of representing the same quantity, as in $2(x + x + 4) = 4x + 8$. Equations may show general relationships as in $t = 1.06p$.

CONNECTIONS

Before working with formal algebra, students should have experience with variables, with functions and their representations, and with the meaning of "equal." In formal algebra students will express general numerical relationships and mathematical structures symbolically and work with the relationships in more complex ways than arithmetic methods. A mathematical structure written with symbols can be operated on. Students will learn to combine like terms, factor, and subtract the same term from both sides of an equation.

While many people do not use formal algebra or functions in their everyday work lives, the concepts embedded in algebra give people tools to consider varying quantities in various situations and to think abstractly.

TEACHING TIPS

Understanding the Foundations of Algebra and Functions

- *Give students early experiences with variables used in general relationships so that variables are not seen as only representing "unknown" or "missing" quantities.*

  For example, students can list all of the possible whole-number (x,y) pairs of values when $x + y = 10$. Also, they can discuss possible values (including non-whole number values) for x and y in $5x = y$, or in $xy = 24$.

5x = y	
x	y
1	5
2	10
0.5	2.5
10	50

- *Provide experiences with data from a relationship that is expressed verbally, in a table, on a graph, and as a symbolic rule (equation).*

  Include examination of data from science class, such as plant growth vs. time data, noting relationships such as, "It grew about 1/4 inch per day from Day 24 to Day 35." Show the data in a table and on a coordinate graph.

  Study the perimeter of squares as the length of the sides change. Note that the total perimeter depends on the length of a side and is related to 4 times the length of a side. Look at the data in a table and on a coordinate graph, and write a simple rule, "perimeter equals 4 times the side length."

- *Develop the concept of a variable by first looking at a variety of numerical replacements, and gradually discussing variables as representing a range of values.*

  For example, $C = 0.25N$ can represent the cost of various numbers of 25 cent candies and can be tested for a variety of whole-number values of N. Students can discuss what range of values of N are possible for this situation.

- ***Give students in-depth experiences with a real-life function relationship between two quantities, and then express the relationship using variables.***

  For example, students can simulate having a candy store where people choose and weigh bags of candy in kilograms and multiply the weight by the price per kilogram to find out the price. A chart of weights and corresponding prices can be made, variables chosen, and the relationship represented to show that the two variables are related by multiplying by the constant cost per kilogram.

w · $3.00 = p	
weight (w)	price (p)
0.5 kg	$1.50
1	3.00
4	12.00
10	30.00

- ***Look for a constant rate of change in the dependent variable when examining data in a table.***

  Notice situations where an increase of 1 for the independent variable is accompanied by a constant increase in the data for the dependent variable. This situation is important because it occurs in linear functions that have straight-line graphs. For example, this chart has an increase of 4 for the second variable for every increase of 1 for the first variable. Extend the table to predict other values. Challenge students to orally, and later symbolically, describe how to obtain the second value (y) when the first value (x) is known. This is sometimes called "What's My Rule?" and involves actually finding the function rule for a set of data.

?	
x	y
0	3
1	7
2	11
3	15

- ***Emphasize that an equal sign expresses a symmetric relationship of equality between two sides of an equation.***

  Elementary mathematics experience may have led students to think that an equal sign signals that an "answer" is coming, or that it says "gives" or "makes." Explore number sentences such as $3 + 1 + 1 = 6 - 1$ to emphasize "is the same as." Look at equal fractions such as $\frac{1}{4} = \frac{16}{64}$ to reinforce the idea that equal quantities can be represented by numbers (or expressions) that do not look the same. Some students will also be ready to recognize that an equation remains a statement of equality when the same amount is added, or subtracted, from each side. For example, $3 + 1 + 1 + 20 = 6 - 1 + 20$ is a true equation, based on the equation above and adding 20 to both sides.

- ***Encourage students to write several possible equations for a situation.***

  For example, the cost of buying sets of party favors for n guests (based on prices in the table) might be re-presented by $15n + 15n + 30n + 40n$, or by $2(15n) + 30n + 40n$, or by $n(15 + 15 + 30 + 40)$. Notice that the first and third expressions together are an example of the distributive property.

All Party Favors On Sale Today!
Balloons............15¢ each
Party Hats.........15¢ each
Bubble Mix........30¢ each
Mini Puzzles .....40¢ each

- ***Look at specific differences between arithmetic and algebraic conventions.***

  In arithmetic, when given a symbolic expression such as 4 + 8, procedures can be followed and an answer can always be found. In algebra, abstract symbolic expressions have meaning as mathematical entities and may not be tied to any concrete problem or specific numeric answer. For example, the expression $x + 4$ names the operation of adding 4 to x, and also describes the quantity "four more than x," for any value of x.

- ***Give informal exposure to the properties of algebra while working on the concept of equality.***

  Students should not only be told that $a + b = b + a$ works for all numbers a and b. They should also be asked "Do you think this is always true? Can you think of a counter-example when it would not be true?" The following properties (without their formal names at first) should be explored in such an inquiry-based manner during the elementary years:

  Commutative properties of addition and multiplication:
  $a + b = b + a$ and $a \times b = b \times a$

  Associative properties of addition and multiplication:
  $a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$

  Distributive property:
  $a \times (b + c) = a \times b + a \times c$

ASSESSMENT ITEM

1. Marlene made 6 batches of muffins. There were 24 muffins in each batch. Which of the following number sentences could be used to find the number of muffins she made?

Response choices	Percent choosing the response
	Grade 4
6 x □ = 24	27
6 ÷ 24 = □	9
6 + □ = 24	3
6 x 24 = □*	37

(1992 NAEP, Grade 4)

applying place value understanding to algebra expressions such as 4x - Students sometimes evaluate $4x$ as 42 when given that $x = 2$.

adding rather than multiplying for expressions such as xy - Students may add the values of the two variables rather than multiplying.

applying mental math problem solving strategies when algebraically representing a situation -
For example, "Carol has some cards, but Joe has 4 more than Carol. If Joe has 30, how many does Carol have?" In arithmetic, students learn to subtract in this "4 more" situation. But the structure and word order of the situation should be linked to $c + 4 = 30$, rather than to $c = 30 - 4$.

adding coefficients and numbers - For example, "If $x = 10$ crayons and the number of crayons $= 4 + 3x$, how many crayons are there?" may mistakenly be solved as $7x = 70$ crayons.

Reading, Writing, and Discussing Algebra

- ***Distinguish between the use of letters as abbreviations for units of measurement and their use as variables.***

  For example, 5m means 5 meters in arithmetic. In algebra, if m stands for meters, it would mean 5 times the *number* of meters.

- ***Explain that any letter may be chosen for a variable, and that choices are made so that the variable is easy to recognize, but any letter may be used.***

  For example, d is often used to represent a distance, but x could be used. Encourage students to write down a "key" such as "d = distance in km" or "x = the number of pounds."

- ***Define variables with specific phrases, such as "Let t equal the number of teachers."***

  Phrases such as "let *b* equal boys" mask the numeric meaning of a variable. The variable *b* might represent the weight of the boys in pounds, or the money earned by the boys, or simply the number of boys.

- ***Reinforce the meaning of the equal sign.***

  It can be read as "equals" or "is the same as." Include experiences with equations with the variable isolated on either the left or the right, such as $x = 25(100 - 1)$ and also equations such as $4^2 = x + 2$, and $x = x$. Let students invent "fancy" ways of expressing a quantity. For example, after starting with $16 = 16$ a student might write $16 = 4 \cdot 4 = (10 - 6)2^2$.

- ***Develop specific strategies for interpreting and understanding the mathematical relationships in algebraic word problems.***

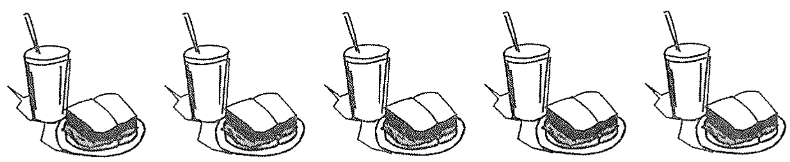
  Summarizing the situation, relating the situation to a familiar situation, identifying relevant data, and identifying variables are all helpful steps to take before writing an equation.

- ***Give students varied experiences translating verbal statements into expressions.***

  Work on relationships such as "more than," "less than," "times as many." $(x + 4)$ represents "four more than x," while $(x - 4)$ represents "four less (or fewer) than x." "4 times as many as x" is written $4x$. So "John has 16 fewer than 3 times as many cards as Kevin" could be written $j = 3k - 16$.

- ***Have students practice reading algebraic statements with words that reflect the meaning of the statement.***

  For example, $n(s + d) = ns + nd$, where n = the number ordered, s = the cost of a sandwich in dollars, and d = the cost of a drink in dollars, can be read as "The number of orders times the cost of a sandwich and a drink together is the same as the number of orders times the cost of one sandwich plus the number of orders times the cost of one drink." This reading gives more attention to the numeric values than "the orders times the sandwiches plus the drinks..."



- ***Reinforce the habit of comparing an equation to its verbal statement and testing it with a single numeric value after writing an equation.***

  For example, "There are 30 students for each teacher" may be incorrectly represented as $30s = t$. But, the mistake can be caught when this equation is read as "the number of students times 30 is the same as the number of teachers," or when a value for s is tested.

Developing Foundations for Computational Facility

- ***Build understanding of the equal sign as a symbol of equivalence between the left and right sides of an equation.***

  Have students make up arithmetic identities (equations without variables) that have one or more operations on both sides, and emphasize that both sides have the same value. For example, a student might write 6 x 8 = 12 x 4, or 6 x 8 + 2 = 12 x 4 + 2, or 6 x 8 + 2 = 10 x 5.

- ***Develop understanding of an unknown in an equation.***

  After students make up arithmetic identities, such as 6 x 8 + 2 = 10 x 5, have them "hide" one number in the identity and later replace the hidden number with a letter. For example, 6 x 8 + 2 = A x 5. Let students try to informally solve such problems.

  6 x 8 + 2 =  **x 5**

- ***Develop understanding of multiple occurrences of a variable in an equation.***

  After students make up single-variable problems, have them "hide" the same number on both the left and right sides of an identity. For example, 6 x 8 + 2 = 25 x 2 can become 6 x 8 + B = 25 x B. It isn't necessary to solve every problem made up, but rather to recognize that a valid equation has been made, even with variables on both sides.

- ***Have students invent and describe procedures, or steps, to follow for specific situations, and have their classmates follow them.***

  For example, for finding the area of any t-shaped room, a student might say "Think of it as two rooms. Measure the length and width of the first room and multiply. Then do the same for the second room. Finally, add together the two areas." Writing verbal statements is a useful skill before writing symbolic equations.

- ***Realize that "guess and check" and "work backward" are not general algebra processes.***

  While students may use these methods to work with their made-up equations, they should look forward to learning general procedures in future math classes (including performing the same operation on both sides and following the order of operations).

ASSESSMENT ITEMS

Item	Percent correct	
	Grade 3	Grade 7
1. 4 + □ = 9	94	—
2. 27 = □ + 14	49	90
3. □ − 8 = 8	24	60

(1986 NAEP, Grades 3, 7)

LEARNING PITFALLS

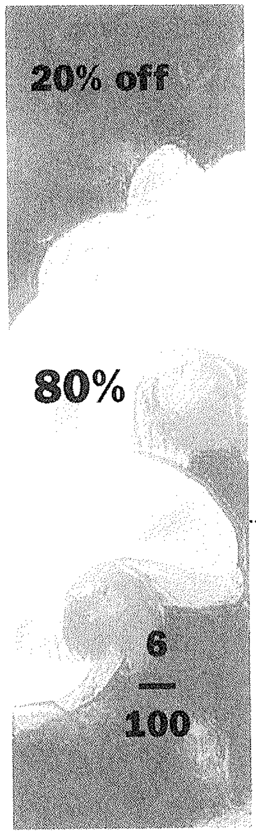
thinking that a problem can have one solution when solved with algebra and a different solution when solved mentally with arithmetic - Students sometimes view algebraic work as detached from other mathematical understandings.

thinking that algebra is a new math exercise for finding answers - Students must develop an understanding that algebra is a way of generalizing arithmetic relationships and communicating those relationships.

not recognizing transformed equations as producing equivalent solutions - For example, $x - 4 = 11$ and $x - 4 + 4 = 11 + 4$ should be recognized as two versions of the same equality.

believing that an equation should have an answer on the right side - Many students have a limited concept of equality and feel that a single value should follow the equal sign.

relying on calculation rather than mathematical structure - When faced with questions of equality, many children must calculate before making a conclusion in situations such as "are these equal: $322 + 523 - 79$ and $79 + 322 - 523$?" and "are these equal: $a + b - c$ and $a - b + c$?"



Percents

Percents are a means of representing or describing a relationship between two quantities in terms of "per 100." Percents are used both to clearly report single statistics, as in "The voter turnout was 35%," and as part of general functional relationships, as in "The sales tax rate is 6%."

KEY CONCEPTS

- *Percents are used for statistical purposes.*

  Percents such as "40% of students have a grade of A," "3% of daily calcium requirements," a road grade of 5%," "10% real juice," and "a 2% probability of rain" serve as reports of relationships between two things, such as the $\frac{\text{measure of real juice}}{\text{measure of the whole drink}}$. When used for statistical purposes, a percent is not used for calculations but rather for communication and comparison. Reporting a ratio as a percent makes it easy to compare to other amounts written as percents because each has a denominator of 100.

- ***Percents work in functional relationships either to scale up or scale down a given quantity, producing a new quantity.***

  Percents are often used to represent general relationships involving multiplication by a factor that is written as a percent. For example, "Give a tip of 20% of the total bill" is a general rule for figuring a tip amount. According to this rule, any amount of a bill in dollars is scaled down to give a suggested tip amount in dollars. Multiplying by a percent greater than 100 results in scaling up. For example, "The prices increased 110%" would lead to the change of a price of $45 to $94.50." Multiplying by 100% gives the original amount.

- *Percents can represent part-whole relationships.*

  For example, "a 35% voter turnout" gives the relationship of voters (the part) to all registered voters (the whole) in terms of the number of voters for every 100 registered voters. The underlying ratio of voters might be 70,000:200,000. This same ratio might also be stated as 7:20, or $\frac{35}{100}$, or 35%.

- *Percents may be used to make comparisons or to note change over time.*

  For example, the voter turnout for two different counties on the same day, or for two different years, can be compared using percents, even though the number of registered voters is different for each place or year. For example, "County A had a turnout of $\frac{17,908}{57,642}$ and County B had a turnout of $\frac{101,920}{395,462}$" presents the data, but leaves comparisons difficult to make. Because all percents are numbers compared to 100, using a percent to express the voter turnout is like using a common denominator. Thus, "County A had a 31% turnout, and County B had a 26% turnout."

CONNECTIONS

Percents are frequently used to report statistics and to make comparisons in social science, chemistry, and in media reports.

Fractions, ratios, rates, proportions, decimals, and percents all have an underlying relationship where two numbers are compared by division.

Understanding percents depends on an understanding of multiplication relationships, in particular, multiplying to expand (or shrink) a quantity by a given factor.

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TEACHING TIPS

Developing Students' Understanding of Percent

- *Develop the concept that two quantities need to be known before establishing a percent.*

  For example, 8 gallons in a 20-gallon tank can be represented by saying "40% full" because both "the original," or total, amount (20 gallons) and the "changed" amount (8 gallons) are known. The ratio 8:20 is associated with the number 40% because it is equivalent to the ratio 40:100. Spend some time predicting the underlying numbers for textbook situations where the percent is given, such as "40% completed free throws." This percent may stem from a ratio of 4:10 or 256:640, etc.

- *Percents represent general constant relationships.*

  The same percent exists in many different instances, for many different base or total amounts. First, two known values (with the same units) can be used to find the constant factor in the percent relationship. Then, the base quantity can vary, as long as the relationship maintains the same constant factor. For example, lemonade that is found to be 30% pure juice because it has 3 gallons of juice in 10 total gallons would also have 30 ounces lemon juice in 100 ounces lemonade. Ten teaspoons of lemonade could be calculated to have 3 teaspoons of juice. A table can help illustrate this idea. Notice this illustrates that many equal ratios or fractions are equal to 30%.

Juice	0.3 liter	1 cup	1.5 drops	3 tsp	30 cans
Total drink	1.0 liter	$3\frac{1}{3}$ cup	5 drops	10 tsp	100 cans

- *Expose students to varied problems that relate two like quantities by percents.*

  For example, the calories from fat can be compared to the calories in an amount of milk by dividing and then converting to a percent. This is an example of comparing a part to a total amount of mixture. Other examples should include comparisons of the amount present to the total amount possible, as in a comparison of dollars saved to dollars needed. Note that all percents stem from two quantities that have the same units. Also, note that if the two quantities are widely different, percents are usually not used in order to avoid small decimals such as 0.0005%. Instead, the amount could be written using scientific notation (5×10^{-6}) or by using "5 parts per million."

- *Give students initial instruction in the relational language of percents before teaching calculation procedures.*

  For example, percents can be tied to known fractions and to decimal hundredths using examples,such as "If $\frac{1}{5}$ of the players are promised a prize, how may prizes *for every* 100 players?" Also, "if 1 of every 5 will win, *at this rate* how many winners will there be out of 100?"

- *Tie common percents to known benchmark fractions using a part-whole model, in addition to exploring the underlying ratios for percents.*

  $1\% = \frac{1}{100}$, $10\% = \frac{10}{100} = \frac{1}{10}$, $25\% = \frac{25}{100} = \frac{1}{4}$, $50\% = \frac{1}{2}$, $75\% = \frac{3}{4}$, $100\% = 1$.

  Area models on grid paper with 100 lightly shaded parts can be used to show equivalence between a percent and a fractional amount such as $\frac{1}{4}$.

- *Teach that one purpose of percents, like ratios, is to make comparisons.*

  Percents give information, such as the amount of money raised compared to the total goal, and also make possible comparisons to other similar situations, as in "Room 10 has raised 40% of its goal, while the entire school has raised 33% of its goal." These percents are easily compared, while ratios with dollar amounts would not reveal the information as clearly.

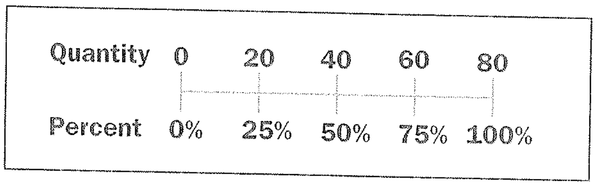
	Earned	Goal	% earned
Buena Vista School	\$1283	\$3850	33%
Room 10	\$70	\$175	40%

- ***Extend the part-whole model to represent both 100 parts and a given total amount other than 1.***

  For example, if the whole amount, all 100 parts, stands for $12000 for a new car, each 1% can be labeled as $120. Thus, a vehicle tax of 6% can be shown to be 6($120).

- ***Work with comparison scales to show various percents related to a quantity.***

  In this example, one whole measure contains 80 units, or actual items, which is equal to 100 out of 100 equal parts. If 80 relates to 100%, 40 relates to 50%, and 20 to 25%. Fractions and decimals can also be represented on such a scale.



- ***Identify the parts of a percent relationship when reading word problems.***

  The base (original) quantity, the percent (rate), and the result of the change are components of every percent relationship.

- ***A percent can be viewed as telling the amount in the whole but also the amount in each part.***

  For example, "If cereal X has 14% of its weight from sugar, how many grams of sugar are in a 30 g serving?" can be solved using a fraction or a scale factor. 14% can be read as "14g for every 100g" or $\frac{14}{100}$ like a fraction. $\frac{14}{100}$ x 30g gives the fraction of the cereal that is sugar. This problem can also be solved by using the scale factor "0.14 g sugar for each part or 1 gram." So, 0.14 can be multiplied times 30g.

ASSESSMENT ITEMS

1. Which of the following is true about 125% of 10?

   A)* It is greater than 10.

   B) It is less than 10.

   C) It is equal to 10

   D) Can't tell.

   E) I don't know.

(1986 and 1990 NAEP, Grades 8 and 12)

2. Which of the following is true about 87% of 10?

   A) It is greater than 10.

   B)* It is less than 10.

   C) It is equal to 10

   D) Can't tell.

   E) I don't know.

(1986 and 1990 NAEP, Grades 8 and 12. In 1986, less than 16% of 13-year-olds were able to perform at or above this level.)

LEARNING PITFALLS

avoiding percents over 100% - Students may try to tie 120% to a part-whole model and say "you can't show more than the whole thing." Thinking of scaling up, or stretching, by a factor would make more sense.

detaching meaning from percents - Students may view percents simply as an alternative way of writing decimals or fractions rather than as special representations of relationships between two quantities. Overemphasis on equality between fraction, decimal, and percent forms masks the many meanings and uses of percents.

thinking percents are only part of 1 after working with a 100% square - As a result, students may not attach meaning to questions such as "What is 35% of 40?"

Reading, Writing, and Discussing Percents

- *Use the language of comparison orally to develop the concept of ratio that is needed for work with percents.*

  Even young children can discuss "3 of these for 4 of those, so 6 of these for 8 of those" and eventually "75 of these for 100 of those." Also, ratios such as "two for you *for every* 5 for me" should be used.

- *Percents give a comparison to a hundred.*

  Historically, they were expressed as "6 dollars (tax) for 100 dollars (earnings)" or "10 pounds (interest) per 100 pounds (borrowed)." 25% can be read "25 percent" and also "25 per hundred."

- *Discuss the common language use of the word "percent."*

  "I'm with you 100%" can be thought of as "I'm with you all the way" or "I'm behind you the whole way." "He gave his 200%" is an exaggeration.

- *Percent notation is often used because it is easier to use in oral language.*

  For example, "fifteen percent" is easier to pronounce than "zero point one five" or "fifteen hundredths."

- *Have students list percent situation ideas and then write and share their own word problems.*

  When reading another student's problem, encourage students to attempt to "read" the problem mathematically. Students should practice identifying the base (original) quantity, the percent, and the result of the change in each problem.

- *Identify the base number (original quantity), the percentage (12%), and the resulting number (result, or outcome) after reading a percent problem.*

  It is often helpful to have students restate a percent word problem to make all components clear. For example, "The absenteeism rate is 12%" implies that "12 *people are absent* for every 100 *people in the total school population.*" For a known population of 1000 students (the original quantity), 12% of the total population (the percentage) of 1000 students is 120 absent students (the outcome or part).

- *Rephrase statements, making the base explicit to clarify what is being compared.*

  For example, "Hawaii depends on Asia for 35% of her tourists" has an implied total number of tourists. The statement can be rephrased as "The number of tourists to Hawaii from Asia is 35% of all of Hawaii's tourists."

- *Point out confusion surrounding the phrase "times more."*

  Some people would say "12 is 4 *times more than 3*" when they mean "12 is 4 times as many as 3." With percents, "4% *more than* 50" would mean "add 4% of 50 to 50 ," which is 52. Similarly, "200% more than before" would mean 300% of the original quantity.

LEARNING PITFALLS

> ***interpreting percents with decimals according to the decimal without consideration of the percent sign*** - Many students and adults are led to think that 0.95% is almost all of something, as when a questionable charity states that 0.95% of each donation goes to the cause.
>
> ***thinking of all percents as part-whole situations*** - Questions such as "What is 150% of 200 hours?" and "What percent of 200 hours is 300 hours?" are difficult to interpret in terms of part of a whole.
>
> ***Confusing "of" and "off"*** - Students sometimes interpret "25% of $160" as the same as "25% off $160."

Developing Computational Facility

- *Begin to develop understanding of the underlying ratios and uses of percents before teaching conversions of percents to fractions and decimals so that percents are understood to be more than alternative ways of writing numbers.*

  Students should connect the use of a percent with its role as a constant factor that can relate two quantities. Students should also view a percent as a representation of an underlying comparison, rather than as a decimal written differently. Equivalencies between fractions, decimals, and percents are not ends in themselves but rather means of preparing for calculation and means of connecting amounts to more familiar forms. For example, it helps to know that 75% of something is the same as $\frac{3}{4}$ of it and that $\frac{4}{5}$ of something is the same as 80% of it.

- ***Converting a percent to a decimal, as might be done before multiplying, should be done with attention to hundredths.***

  For example, 4% is 0.04, and 12.5% is 0.125. In both cases, the original numeral from the ones place in the percent ends up in the hundredths place. This is sometimes explained by noting that having a percent, or "out of 100," of a number is like dividing the number by 100, and dividing by 100 causes the decimal point to shift two places to the left.

- ***Think about "times 100" and "divide by 100" when converting a decimal to a percent.***

  Writing a decimal number as an equivalent percent requires multiplying by 100 before writing a % sign, because the percent sign means divide by 100, or one hundredth of the amount. Thus 0.4 can be thought of as

  $$0.4 = 0.4 \times 100 \div 100 = 40 \div 100 = 40\ \%.$$

  0.4 and 40% are equal because any number multiplied by 100 and then divided by 100 is the same number.

- ***Continue to emphasize the connection to 100 when converting a fraction to a percent.***

  $$\boxed{\frac{3 \times 20}{5 \times 20} = \frac{?}{100}}$$

  Before using the algorithm "divide numerator by denominator, then move the decimal to the right two places and put on a % sign," use familiar procedures for finding equivalent fractions. In the example, the fraction $\frac{3}{5}$ "needs" to be multiplied by a factor to get a denominator of 100, so divide 100 by 5 and multiply that result (20) by the numerator and the denominator.

- ***Teach how to find 1% and 10% of a quantity.***

  Students should be able to mentally find 1% of a number, because it can be based on understanding of multiplication by a scale factor of 0.01. By looking at the results of several calculations, students can see that 10% is not only 10 times whatever 1% is, but it is also one-tenth (or 0.1) of the number. It can then follow that 23% of a number is both 10% + 10% + 1% + 1% + 1% of the number and 23 times the 1% amount. This method will help students who experience temporary difficulty remembering the algorithm involving multiplying and moving the decimal.

23% of \$600 =	**\$60** +	**\$60** +	**\$6** +	**\$6** +	**\$6**
	(10%) +	**(10%)** +	**(1%)** +	**(1%)** +	**(1%)**

- *Teach estimation before teaching an algorithm for multiplying so that computation problems can be checked for reasonable answers.*

  Have students make rough estimates by using benchmark fractions: $\frac{1}{100}$, $\frac{1}{10}$, $\frac{1}{4}$, $\frac{1}{2}$, 1. Oral problems can be given "ballpark estimates," as when you hear "she paid only 29% of the real price of the $200 coat" you think "about $\frac{1}{4}$ of the price," or "about $50."

- *Emphasize that both a percent and its decimal form have a base of 100 as part of the comparison.*

  Thus 5% of 200 involves division by 100 and multiplication by 5, just as 0.05(200) can be viewed as finding $\frac{1}{100}$ of 200 and then multiplying by 5.

- *Realize that percent calculation problems fall into three types.*

  "Find a percent of a number" is generally easier because it involves multiplication. "Find the base" involves division to find the original number. "What percent of X is Y?" can be solved by dividing two quantities in the correct order to know the percent. This last problem type can be made easier by preliminary work developing understanding of the underlying ratios for percents.

- *Relate finding percents of numbers, including those involving decimal points in the percent, to everyday situations.*

  For example, a sales tax rate of 6.5% means "pay 6 cents and a half penny for every dollar in the price." Build a table of tax, first for every penny of price and then in increments of a dollar to observe the effect of the "extra" half percent. This also reinforces the functional relationship between the tax rate and the sales tax amount.

- *Connect finding the percent that relates two quantities to finding the factor that will change the original quantity to the outcome.*

  For example, "What percent of 32 is 8?" can be understood as asking "What do you multiply 32 by to get 8?" The factor $\frac{1}{4}$ can be expressed as 25%.

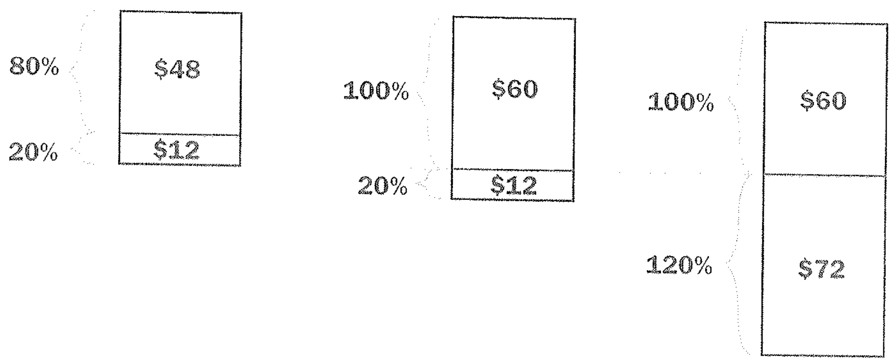
  $\boxed{\frac{?}{100} \cdot 32 = 8}$ $\boxed{? \cdot 32 = 8}$

- *Point out the role of division in finding the original base number when the percent rate and outcome are known.*

  For example, "A car cost $9000 when on sale at 75% of the original price; what is the original price?" can be solved by thinking "what number times 75% equals $9000?" and dividing 9000 by 0.75. Writing an equation 75% • ___ = $9000 emphasizes that 75% is serving as a scale factor to change the original price to $9000.

- ***Discuss relationships that help change an addition or subtraction (increase or decrease) percent situation into a multiplication situation.***

For example, "20% off the price" leaves (100% - 20%) or 80% of the price to be paid. "Add 20% to the price" results in a price that is 120% of the original. "An increase of 120% in price" gives a price that is 220% of the original.



- ***Start with simple numbers when introducing percent increase or decrease problems.***

For example, 6% tax on $200 is $200 + $12, or 106% of $200. Also, an increase of 150% on $500 can be thought of as $500 + $250 in addition to the original $500, or as 250% of $500. Money examples work particularly well because $100 or $1.00 can easily be imagined as having 100 parts.

ASSESSMENT ITEMS

1. Ken bought a used car for $5,375. He had to pay an additional 15 percent of the purchase price to cover both the sales tax and extra fees. Of the following, which is closest to the total amount Ken paid?

Response choices	Percent choosing the response	
	Grade 8	Grade 12
$ 806	25	20
$5,510	14	4
$5,760	12	3
$5,940	7	3
$6,180*	40	69

(1992 NAEP, Grades 8, 12)

2. If the price of a can of beans is raised from 60 cents to 75 cents, what is the percent increase in price?

A) 15% B) 20% C)* 25% D) 30%

(1996 TIMSS, Grades 7, 8)

over-generalizing changing percents to decimals - Experiences with two-digit percents such as 55% = 0.55 can lead to the erroneous rule "just drop the percent sign and put a decimal in front" and thus 120% becomes 0.120, and 0.9% may become 0.9 or even 9.

applying multiplication table knowledge - 8 = _% of 32 often elicits an answer of 4%.

ignoring the percent sign - $\frac{1}{2}$ % of 40" may become 20 and 6% of $50 may become $300.

misusing the "move the decimal" rule - Students sometimes find 1% of a number by moving the decimal one place to the left "because 1 has one digit" and move the decimal 2 places to the left to find 10%, "because 10 has two digits." Students sometimes almost randomly choose whether to move the decimal left or right.

reverting to subtraction for comparisons - A problem like "Bill hiked 6 miles of his 9-mile hike before lunch; what percent is that?" may result in subtraction. Students may incorrectly answer "3%."

using the wrong operation, sometimes based on the "looks" of the numbers and sometimes based on reading "of" - Some students add and some multiply when given "What percent of $80 is $50?" Thus $130, or 4000%, and sometimes 40% are possible incorrect responses.

discomfort with 100% = 1 and with 100% of a quantity - The quantity 1 may seem like a small amount to students who also think of 100% as "everything" or "all of something." Thus, $\frac{9}{9}$ (i.e., 1) of a number may be given as 1% of the number rather than the whole number.

using proportions for all problems - Students often confuse the entries, resulting in errors, unless attention is strictly paid to the underlying relationship.

assuming that decreasing an amount by a percent and then increasing the result by the same percent will give the original amount - For example, 5% off of $1000 gives $950, then adding 5% tax will yield a final price below $1000 because the base has shifted from $1000 to $950.